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Mathematical Reviews

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Vol. 9, No. 3 March, 1948 pp. 129–168

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MATHEMATICAL REVIEWS

Published monthly, except August, by

The American Mathematical Society, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

The American Mathematical Society
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The Institute of Mathematical Statistics
The Edinburgh Mathematical Society
L'Intermédiaire des Recherches Mathématiques
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Het Wiskundig Genootschap te Amsterdam
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Polish Mathematical Society
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Editorial Office

Mathematical Reviews, Brown University, Providence 12, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to Mathematical Reviews. Subscriptions should be addressed to Mathematical Reviews, Lancaster, Pennsylvania, or Brown University, Providence 12, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Naval Research, Department of the Navy, U.S.A. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Mathematical Reviews

Vol. 9, No. 3 MARCH, 1948 Pages 129-168

FOUNDATIONS

Csillag, Paul. **Eine Bemerkung zur Auflösung der eingeschachtelten Rekursion.** Acta Univ. Szeged. Sect. Sci. Math. 11, 169–173 (1947).

R. Péter [Math. Ann. 110, 612–632 (1934)] showed that various types of recursive definitions of numerical functions can be reduced to the "primitive recursive" type $\phi(0, a)=\alpha(a)$, $\phi(n+1, a)=\beta(n, a, \phi(n, a))$, in combination with substitutions. The most difficult of these types was the "nested recursion" (eingeschachtelte Rekursion) where the definiens for $\phi(n+1, a)$ is constructed from previously defined functions and $\phi(n, a)$ with the help of substitutions for the parameter a. Péter's proof for this case was simplified in her thesis [Mat. Fiz. Lapok 42, 25–49 (1935)]. The present posthumous paper gives a further simplification in this proof. It was prepared by Péter from manuscripts left by the author. *H. B. Curry* (State College, Pa.).

Surányi, János. **Zur Reduktion des Entscheidungsproblems des logischen Funktionenkalküls.** Mat. Fiz. Lapok 50, 51–74 (1943). (Hungarian. German summary)

This review is based on the German summary. The author proves the following two theorems relating to the first order predicate calculus. (I) Every formula of the calculus is equivalent, in regard to satisfiability (Erfüllbarkeit) to one in prenex normal form with only binary predicate-variables and a prefix of one of the two forms $(x_1)(x_2)(x_3)(Ey)$, $(x_1)(x_2)(Ey)(x_3)$. (II) Any such formula is also equivalent, in the same sense, to one of prenex form with prefix $(x_1)(x_2)\cdots(x_n)(Ey)$ in which there is only a single binary predicate-variable. *H. B. Curry* (State College, Pa.).

Kalmár, László. **Zielsetzungen, Methoden und Ergebnisse der Hilbertschen Beweistheorie.** Mat. Fiz. Lapok 48, 65–119 (1941). (Hungarian. German summary)

[Reviewed from the German summary.] This is the elaboration of part of a lecture delivered before the Colloquium of the Institute of Theoretical Physics of the University of Budapest. It is a general summary of the Hilbert program: its background, objectives, methods and significance. There is considerable explanatory matter concerning the axiomatic method and the elements of mathematical logic (classical truth tables and first order predicate calculus), and the definitions of "proof," "consistency," etc. The author discusses and coordinates the methods of J. König, Ackermann, von Neumann, Herbrand, Gentzen and Gödel.

H. B. Curry (State College, Pa.).

Péter, Rózsa. **Die Schranken der axiomatischen Methode.** Mat. Fiz. Lapok 48, 120–143 (1941). (Hungarian. German summary)

[Reviewed from the German summary.] This is a report of a lecture delivered before the Colloquium of the Institute of Theoretical Physics of the University of Budapest. It is a complement to the lecture by Kalmár reviewed above. The subject is the incompleteness theorems of mathematical logic. The author discusses various kinds of completeness and reports on the incompleteness theorems of Löwenheim, Skolem, Gödel and Church. *H. B. Curry.*

Mostowski, Andrzej. **On undecidable propositions in formalized systems of mathematics.** Kwartalnik Filozoficzny 16, 223–277 (1946). (Polish)

The aim of the article is to give "a popular, and at the same time completely rigorous, presentation of Gödel's ideas: his theory reaching the conclusion that in every formal system of mathematics, satisfying certain very general conditions, there exist statements such that neither their truth nor falsity can be established within the system." This aim is achieved most successfully. The author presents the proof for a part of Gödel's original system only: this allows him to simplify and mathematicize, in a more classical sense, the logical formalism without impairing the exhibition of the essential ideas.

The paper can serve as an excellent introduction to modern theories of formal systems in general. The simplicity of presentation is such as to suggest implicitly, in the reviewer's opinion, the possibility of algebraization of logical systems including a treatment of quantifiers as mathematical operators. The paper ends with indications of how to extend the results on existence of undecidable propositions from the primary system considered to more general ones dealing also with variables of higher type. *S. M. Ulam.*

Mostowski, Andrzej. **On definable sets of positive integers.** Fund. Math. 34, 81–112 (1947).

The paper is concerned with generalisations of the Gödel-Church-Kleene-Turing general recursive functions and sets, modelled on the construction of projective sets from Borel sets. As the author points out in an added note, some of his results are anticipated by S. C. Kleene in a paper [Trans. Amer. Math. Soc. 53, 41–73 (1943); these Rev. 4, 126] which was not available to him when writing.

If S is a logical system containing elementary number theory, ϕ is decidable in S if for each $\mathfrak{n}_k$ either $\phi(\mathfrak{n}_k)$ or $\phi'(\mathfrak{n}_k)$ ("not $\phi(\mathfrak{n}_k)$") is provable in S, where $\mathfrak{n}_k$ is any element of R_k, the set of all k-tuples of natural numbers. The classes $P_n^{(k)}$ and $Q_n^{(k)}$ are defined inductively on n: $A \varepsilon P_0^{(k)} (= Q_0^{(k)})$ if a decidable ϕ exists, such that $\mathfrak{n}_k \varepsilon A$ if and only if $\phi(\mathfrak{n}_k)$ is provable in S; $A \varepsilon P_{n+1}^{(k)}$ if B of $Q_n^{(k+1)}$ exists, such that $\mathfrak{n}_k \varepsilon A$ if and only if $(\mathfrak{n}_k, p) \varepsilon B$ for some p of R_1; $A \varepsilon Q_{n+1}^{(k)}$ if $R_k - A \varepsilon P_{n+1}^{(k)}$. Next, a class C_s of logics S is defined for each $s=0, 1, 2, \cdots$, by four conditions, of which one requires that $\Delta \varepsilon P_s^{(2)}$, where Δ is the sets of pairs (q, n), such that n and q are the Gödel-numbers respectively of a formula and its proof in $S \varepsilon C_s$; and the other three conditions are similar. Principia Mathematica and all "ordinary" logics are in C_0. It is stated that the "infinite logics" of Rosser

[J. Symbolic Logic 2, 129–137 (1937)] are not in C_0, but each is in C_s for some s.

Some theorems: If $k \geqq 1$ and $S \varepsilon C_s$ for some s, then $P_n^{(k)} \neq P_{n+1}^{(k)}$, $Q_n^{(k)} \neq Q_{n+1}^{(k)}$, $P_n^{(k)} \neq Q_n^{(k)}$, for all $n \geqq 0$. Gödel-theorem: If $S \varepsilon C_s$ for some s and is ω-consistent it is incomplete. The Rosser improvement on this result ("consistent" for "ω-consistent" [J. Symbolic Logic 1, 87–91 (1936)] does not hold for $s > 0$: a counter-example is given. Post's analogue [Bull. Amer. Math. Soc. 50, 284–316 (1944); these Rev. 6, 29] of Suslin's theorem, $P_1^{(k)} \cap Q_1^{(k)} = P_0^{(k)}$, is obtained. Classes $P_n^{k,l}$, $Q_n^{k,l}$ of functions $f: R_k \to R_l$ are defined by the rule that f is in $P_n^{k,l}$ (or $Q_n^{k,l}$) if $\mathfrak{m}_l = f(\mathfrak{n}_k)$ implies $(\mathfrak{n}_k, \mathfrak{m}_l) \varepsilon P_n^{(k+l)}$ (or $Q_n^{(k+l)}$); and $P_0^{(k,1)}$ is proved to be the set of general recursive functions if S satisfies suitable conditions. [On p. 101, paragraph 1, read ϕ' for ϕ, twice.]

M. H. A. Newman (Manchester).

Popper, K. R. New foundations for logic. Mind 56, 193–235 (1947); errata, 57, 69–70 (1948).

This is an exposition, by means of successive tentatives, of a theory of inference which reminds one of the systems given by G. Gentzen, S. Jaskowski and O. Ketonen, but is also influenced by Tarski's well-known work on the concept of logical consequence. This theory is not dependent on logical axioms; the difficulty, mentioned by Tarski, of distinguishing between formative (or logical) and descriptive (or extra-logical) signs, is removed. An inference is called valid if every interpretation which makes the premises true also makes the conclusion true; we can also say: an inference is valid if no counter-example exists. The expression "$a_1, \cdots, a_n/b$" is used in order to express the assertion: "From the statements $a_1, a_2, \cdots, a_n$, the statement b can be derived." The notion of mutual deducibility can be defined as follows: "$a//b$ if, and only if, a/b and b/a."

A number of valid rules of inference are mentioned. The system consisting of the rules "$a_1, \cdots, a_n/a_i$ $(1 \leqq i \leqq n)$" and

$$\text{"if } \left\{\begin{array}{l} a_1, \cdots, a_n/b_1 \\ \text{and } a_1, \cdots, a_n/b_2 \\ \cdots \quad \cdots \quad \cdots \\ \text{and } a_1, \cdots, a_n/b_m \end{array}\right\} \text{ and if } b_1, \cdots, b_m/c, \text{ then } a_1, \cdots, a_n/c\text{"}$$

as well as some other systems, is sufficient as a basis for the construction of propositional and functional logic.

The logic of compound statements is based on a number of primitive rules of derivation. Conjunction, for instance, is characterized by the rule "$a \wedge b/c$ if, and only if, $a, b/c$." It may, however, also be introduced by means of the explicit definition "$a//b \wedge c$ if, and only if, for any c_1: a/c_1 if and only if $b \wedge c/c_1$." The theory of quantification is said to be too subtle to be analysed in detail in this paper. A more systematic and more complete exposition of the system is to be given elsewhere. The paper concludes by some considerations on derivation and demonstration and on meta-language and object language.

E. W. Beth.

***Gokieli, L. P. K Probleme Aksiomatizacii Logiki. [On the Problem of the Axiomatization of Logic].** Izdatel'stvo Akademii Nauk Gruzinskoĭ SSR, Tbilisi, 1947. iii+86 pp.

The author tries to prove by philosophical arguments that a formalization of logic is impossible. The book has no mathematical content.

A. Heyting (Amsterdam).

***Gokieli, L. P. Matematičeskie Rukopisi Karla Marksa i Voprosy Obosnovaniya Matematiki. [The Mathematical Manuscripts of Karl Marx and Questions of the Foundations of Mathematics].** Izdatel'stvo Akademii Nauk Gruzinskoĭ SSR, Tbilisi, 1947. iv+111 pp.

Foster, Alfred L. Natural orderings. I. Univ. California Publ. Math. (N.S.) 1, 147–158 (1944).

The objective motivating this paper is "to study [the] arithmetic [of positive integers] so as to make multiplication . . . basic [instead of addition] and only sparingly . . . admit new operations . . . until it is definable." According to the author, this paper "represents a small step in this direction." Actually, he is concerned only with "natural orderings." Let N_k be the set of nonnegative lattice vectors in Euclidean k-space E_k, where k is any positive integer, a nonnegative lattice vector being a point $(n_1, \cdots, n_k)$ of E_k with nonnegative integral coordinates. A linear order of N_k is a natural ordering of N_k if the order relation $\dashv$ has the property that $\rho \dashv \sigma$ implies $\rho + \tau \dashv \sigma + \tau$ for 3 elements ρ, σ, τ of N_k; here $+$ means ordinary vector addition. The principal result shows that every natural ordering of N_k is representable in terms of special natural orderings, defined by means of projections upon certain specialized mutually orthogonal vectors.

H. Blumberg (Columbus, Ohio).

***Keyser, Cassius Jackson. Mathematics as a Culture Clue and Other Essays.** Scripta Mathematica, New York, N. Y., 1947. vii+277 pp. (4 plates). $3.75.

This book, published as volume I of the author's collected works, consists of reprints of essays on the nature, foundations, cultural value and history of mathematics, which have, as far as I see (there is no mention of place or time of earlier publication), appeared before in Scripta Math. It should prove interesting and profitable reading for anyone interested, professionally or not, in mathematics. I should like to indicate particularly "A glance at some of the ideas of Charles Sanders Peirce."

E. W. Beth (Amsterdam).

***von Neumann, John. The mathematician.** The Works of the Mind, Edited for the Committee on Social Thought by Robert B. Heywood, pp. 180–196. The University of Chicago Press, Chicago, Ill., 1947. $4.00.

White, Leslie A. The locus of mathematical reality: an anthropological footnote. Philos. Sci. 14, 289–303 (1947).

Bergmann, Gustav. The logic of quanta. Amer. J. Phys. 15, 397–408 (1947).

THEORY OF GROUPS

Szép, J. On finite groups which are necessarily commutative. Comment. Math. Helv. 20, 223–224 (1947).

Let $p_1, \cdots, p_n$ be distinct primes and let p_i belong to the exponent $\gamma_{ik} \bmod p_k$. The author proves that if G is a solvable group of order $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ all of whose Sylow subgroups are Abelian, and if $\alpha_i < \min(\gamma_{ik})$ for every i and all $k \neq i$, then G is Abelian. The proof given implies the following: if G is a solvable group of order $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ with $\alpha_i < \min(\gamma_{ik})$ for every i and all $k \neq i$, then G is nilpotent.

S. A. Jennings (Vancouver, B. C.).

Rédei, L. Das "schiefe Produkt" in der Gruppentheorie mit Anwendung auf die endlichen nichtkommutativen Gruppen mit lauter kommutativen echten Untergruppen und die Ordnungszahlen, zu denen nur kommutative Gruppen gehören. Comment. Math. Helv. **20**, 225–264 (1947).

A finite group G is said to be of the first kind (Stufe) if it is noncommutative, but all its proper subgroups are commutative. Such groups were first studied by G. A. Miller and H. C. Moreno [Trans. Amer. Math. Soc. **4**, 398–404 (1903)], and a generalization of them by O. Schmidt [Rec. Math. [Mat. Sbornik] **31**, 366–372 (1924)]. The author gives a complete determination of these groups in terms of what he calls skew products of a group G and a ring R. These are of three types. Type GR. Assuming R has an identity element, let $\alpha\rightarrow\bar{\alpha}$ be a homomorphism of G into the (multiplicative) group of units of R. Then GR consists of all ordered pairs (α, a), α in G and a in R, with product defined by $(\alpha, a)(\beta, b)=(\alpha\beta, a+\bar{\alpha}b)$. Type $G(+)R$. Let $\alpha\rightarrow\alpha'$ be a homomorphism of G into the additive group of R such that $\alpha'\beta'=0$ for all α and β in G. Then $G(+)R$ consists of all ordered pairs (α, a) with product $(\alpha, a)(\beta, b)=(\alpha\beta, a+b+\alpha'b)$. Type $G(\ddagger)R$. Let $\alpha\rightarrow\alpha'$, $\alpha\rightarrow\alpha''$ be two homomorphisms of G into the additive group of R. Then $G(\ddagger)R$ consists of all pairs (α, a) with product $(\alpha, a)(\beta, b)=(\alpha\beta, a+b+\alpha'\beta'')$.

The first two types are extensions of the additive group R^+ of R with factor group G and factor set 1. The third is a central extension of R^+ with factor set $f(\alpha, \beta)=\alpha'\beta''$. Every group of the first kind (Stufe), with the exception of the quaternionic group of order 8, is shown to be one of the following three types. First type. For any primes p, q and any integer $u\geqq 1$, $G_{\mathrm{I}}(p, q, u)=GR$ with G a cyclic group of order p^u and R the finite field having q^v elements, where v is the least positive integer for which $q^v\equiv 1 \pmod{p}$. Second type. For any prime p, and any integers $u\geqq 1$, $v\geqq 2$, $G_{\mathrm{II}}(p, u, v)=G(+)R$ with G a cyclic group of order p^u, and R the finite ring of integers mod p^v. Third type. For any prime p, and any integers $u\geqq v\geqq 1$, $G_{\mathrm{III}}(p, u, v)=G(\ddagger)R$ with G the direct product of two cyclic groups of orders p^u and p^v, and R the finite field of integers mod p. In all three of these, the kernels of the homomorphisms $\bar{\alpha}$, α', α'' must have index p in G, and those of α', α'' must be different. Subject to these requirements, the groups G_{I}, G_{II}, G_{III} do not depend on the choice of the homomorphisms, and are all distinct with the exception $G_{\mathrm{II}}(2, 1, 2)=G_{\mathrm{III}}(2, 1, 1)$.

Several different expressions of these groups are given, including by generating relations. For each such group a catalog is made of its center, commutator subgroup, Sylow subgroups, its automorphisms and the order and number of conjugates of its various elements.

The order of G_{I} is p^uq^v, and a G_{I} exists for this order if and only if $q^v\equiv 1 \pmod{p}$ or $p^u\equiv 1 \pmod{q}$. The orders of G_{II} and G_{III} are, respectively, p^{u+v} and p^{u+v+1}, and for a given order p^e with $e\geqq 3$ there exist $e-2+[\frac{1}{2}(e-1)]$ such groups (none if $e\leqq 2$). Combining these results with the observation that every finite noncommutative group must contain a subgroup of the first kind, the author gives a quick proof of Szép's theorem [see the preceding review] and the following. Let $n=p_1^{a_1}p_2^{a_2}\cdots$ be the decomposition into primes of an integer n, and define $\Phi(n)=(p_1^{a_1}-1)(p_2^{a_2}-1)\cdots$. Then n has the property that every group of order n is commutative if and only if (1) no exponent e_i exceeds 2 and (2) $\Phi(n)$ is prime to n. *A. H. Clifford* (Baltimore, Md.).

Szele, T. Über die endlichen Ordnungszahlen, zu denen nur eine Gruppe gehört. Comment. Math. Helv. **20**, 265–267 (1947).

A corollary of Rédei's theorem on the orders n for which only commutative groups can exist [see the end of the preceding review] is the following. An integer n has the property that there exists only one group of order n (naturally the cyclic one) if and only if it is prime to (Euler's) $\phi(n)$. The author gives a simple direct proof of this using only a single theorem of Frobenius. *A. H. Clifford.*

Szele, T. Ein Beweis des Ruffini-Abelschen Satzes. Comment. Math. Helv. **20**, 268–269 (1947).

A very short proof is given at the same time for the two facts that the alternating group of more than 4 elements has no self-conjugate subgroup of prime index and that the symmetric group has no such subgroup apart from the alternating group. The proof is based on the fact that a self-conjugate subgroup with Abelian quotient group contains the commutator subgroup. It follows that the commutator of (abc) and $(ab)(de)$ which is again (abc) lies in the self-conjugate subgroup, hence this must be the alternating group. *O. Todd-Taussky* (Princeton, N. J.).

Vincent, Georges. Les groupes linéaires finis sans points fixes. Comment. Math. Helv. **20**, 117–171 (1947).

Burnside [Messenger of Math. **35**, 46–55 (1905)] discussed the properties of groups whose Sylow subgroups are (1) all cyclical, (2) cyclical for $p\neq 2$, of quaternion type for $p=2$. The groups of type (1) are soluble while those of type (2) may or may not be soluble. He proved the following theorem. If $\mathfrak{G}$ is a finite group and Γ a faithful representation of $\mathfrak{G}$ as an irreducible group of unitary matrices, then, unless $\mathfrak{G}$ is of one of the above types, there is at least one operation of $\mathfrak{G}$ different from the identity whose corresponding matrix in Γ has at least one characteristic root equal to 1.

Hopf [Math. Ann. **95**, 313–339 (1925)] pointed out the topological significance with regard to the classification of Clifford-Klein models of those groups of rotations of a sphere S^n, no transformation of which other than the identity leaves a point of S^n fixed, i.e., has a characteristic root equal to 1. The present paper is written from this topological point of view and seeks to determine all faithful orthogonal representations of Burnside's groups of types (1) and (2) which are irreducible in the real field. The author is successful so far as groups of type (1) are concerned, but groups of type (2) offer greater difficulties and his conclusions are confined to groups of class (α), i.e., groups $\mathfrak{G}$ with cyclical commutator subgroup $\mathfrak{G}'$ defined by the relations

$$A^{2^{\alpha-1}n}=I,\quad B^{2^u}=A^{2^{\alpha-2}n},\quad BAB^{-1}=A^r,$$

with certain numerical conditions on α, n, u, r.

The paper begins with an alternative proof of Burnside's theorem and a classification of groups of type (1). In considering groups of type (2) Burnside oversimplified the problem [loc. cit., p. 50]. Utilizing Schreier's theory of group extension the author is able to pass from a cyclical $\mathfrak{G}'$ whose order is divisible by $2^{\alpha-2}n$, with n odd, to $\mathfrak{G}$ where $\mathfrak{G}/\mathfrak{G}'$ is Abelian of type $(2, 2, u)$, with u odd. Several illustrative examples are given.

In seeking irreducible unitary representations of degree x without fixed points, it is important to distinguish those which are orthogonal from those which are complex. The latter category lead to orthogonal representations of degree $2x$ which are irreducible in the real field. The necessary and

sufficient condition for the existence of a representation having no fixed points of a noncyclical group $\mathfrak{G}$ of type (1) of order $g=mdn'$ defined by the relations $A^m=I$, $B^{dn'}=I$, $BAB^{-1}=A^r$, where n', r, $r-1$, d are all prime to m, is that n' and d should have the same prime factors, though the multiplicities need not be the same [Burnside, loc. cit.]. Under such circumstances all the irreducible faithful representations of $\mathfrak{G}$ are free from fixed points, are of the same degree d, and in number $\varphi(g)/d^2$; these representations are complex with real or complex character according as $n'=2$ or $n'>2$. For groups of type (2), class (α), a similar theorem holds.

Turning to the applications, it is well known that there are only two finite rotation groups without fixed points of the sphere S^{2k}, namely the identity and a cyclical group of order 2. The author concludes that the finite rotation groups without fixed points of a sphere S^{4k+1} are all of type (1), while those of a sphere S^{4k+3} may be of type (1) or (2). Since these two types exhaust the possibilities, it follows that the fundamental group $\mathfrak{G}$ of any Clifford-Klein model is either of type (1) or (2) and the first Betti group $\mathfrak{G}/\mathfrak{G}'$ is known. *G. de B. Robinson* (Toronto, Ont.).

Sadowski, L. On structural isomorphisms of free products of groups. Rec. Math. [Mat. Sbornik] N.S. **21(63)**, 63–82 (1947). (Russian. English summary)

In an earlier paper [same Rec. N.S. **14(56)**, 155–173 (1944); these Rev. **6**, 201] the author studied the structural (lattice) isomorphisms of free groups. Here the work is extended to free products. The major part of the paper leads to the following lemma. If between $G=\{a\}*\{b\}$ and G' there is a structural isomorphism f, $G'=G'$, then for every pair of generatrices a, b of the cyclic groups $\{a\}$ and $\{b\}$ we may find generatrices a' and b' of $\{a'\}=\{a\}^f$ and $\{b'\}=\{b\}^f$ such that $\{ab\}^f=\{a'b'\}$. From this he proves his main theorem: a group G decomposable as a free product $G=A*B$ is determined by its subgroup structure. It remains an open question whether or not every lattice automorphism of G is induced by a group automorphism, as he proved for all free groups except the infinite cyclic group. *M. Hall, Jr.* (Columbus, Ohio).

Lombardo-Radice, L. Il difetto di regolarità di un gruppo finito rispetto a un divisore primo del suo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 766–767 (1947).

The author announces some results on modular group rings. The proofs are to appear in a later publication. *I. Kaplansky* (Chicago, Ill.).

Eilenberg, Samuel, and MacLane, Saunders. Algebraic cohomology groups and loops. Duke Math. J. **14**, 435–463 (1947).

Let $H^n(Q, G)$ be the n-dimensional cohomology group of a group Q with Abelian coefficient group G [see the authors' papers: Ann. of Math. (2) **48**, 51–78, 326–341 (1947); these Rev. **8**, 367; **9**, 7]. The authors develop for $n\geqq 3$ a purely algebraic interpretation for $H^n(Q, G)$ analogous to the known interpretation for $H^2(Q, G)$ in terms of the central group extension (E, ϕ) of Q by G. For this purpose the extensions (E, ϕ) are replaced by "loop prolongations" (L, ϕ) of Q by G. Here ϕ is a homomorphism of the loop L onto Q and the kernel K of ϕ contains, but may be larger than, G. Only those prolongations are considered which satisfy certain conditions such, for example, as being "central." Several of the conditions are stated in terms of associators, namely, elements of L which measure the deviation from the associative law just as commutators in a non-Abelian group measure the deviation from commutativity. The associator $A(a, b, c)$ (a, b, c in L) satisfies an identity which parallels the condition that a function of three variables in Q is a 3-cocycle. This leads to the establishment of a mapping of the prolongations of Q by G onto the 3-dimensional cocycles of Q. The allowable prolongations fall into certain similarity classes and, with a suitable product for prolongations, the mapping in question induces an isomorphism between classes of prolongations and the group of 3-dimensional cocycles of Q. The cocycles can be replaced by 3-dimensional cohomology classes if similarity is suitably modified. The higher dimensional cohomology groups can be realized in like manner with the aid of iterated associators such as $A(a, b, A(c, d, e))$, but the realization of cocycles carries through only when n is odd. *P. A. Smith.*

Dynkin, E. B. Calculation of the coefficients in the Campbell-Hausdorff formula. Doklady Akad. Nauk SSSR (N.S.) **57**, 323–326 (1947). (Russian)

The author points out that no effective procedure has existed hitherto for the effective calculation of the polynomials $P_m(x, y)$ appearing in the Campbell-Hausdorff formula [J. E. Campbell, Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups, Oxford, 1903; F. Hausdorff, Ber. Verh. Sachs. Ges. Wiss. **58**, 19–48 (1906)]. He proceeds as follows. Let K be an arbitrary field of characteristic zero. The system R of all noncommutative polynomials over K in a finite number of the marks $x_1, x_2, \cdots$, is a free associative algebra over K with these marks as generators. Let R^0 denote the minimal subsystem of R containing the basic marks, and such that if the polynomials P and Q belong to R^0 then so also do both $aP+bQ$ ($a, b \varepsilon K$) and the commutator $PoQ=PQ-QP$. The linear mapping defined by an extension of the correspondence

$$x_{i_1}x_{i_2}\cdots x_{i_k}\rightarrow(\cdots(((x_{i_1}ox_{i_2})ox_{i_3})o\cdots)ox_{i_k})$$

maps each polynomial P of R into a polynomial P^0 of R^0. The author proves the theorem: if $P \varepsilon R^0$ then $P^0=P$. From this theorem he obtains a solution of his problem and a new proof of the Campbell-Hausdorff formula. *L. Zippin.*

Harish-Chandra. Infinite irreducible representations of the Lorentz group. Proc. Roy. Soc. London. Ser. A. **189**, 372–401 (1947).

The paper starts with the determination of the operators which satisfy the commutation relations of the infinitesimal operators of the homogeneous proper Lorentz group. This leads to an enumeration of all the one or two-valued representations of this group. [The unitary representations were recently completely determined by V. Bargmann [see the following review] and by L. Gelfand and M. Neumark [reference at the end of the following review]. These writers give not only the infinitesimal operators but also the operators for finite Lorentz transformations.] The existence of infinitesimal operators is assumed in the present article. The further assumption is made that no irreducible representation of the Lorentz group contains more than once any irreducible representation of the subgroup formed by spacelike rotations. It is stated, however, that this assumption can be avoided and this is corroborated also by Bargmann's work.

The result is as follows: an irreducible representation corresponds to every pair of complex numbers κ, κ' the difference of which is a real integer or half integer. The

κ, κ' can be expressed by the fundamental invariants J^2 and I [$-Q$ and R in Bargmann's notation], $J^2-2iI=4\kappa(\kappa+1)$ and $J^2+2iI=4\kappa'(\kappa'+1)$. The J^2 is the sum of the squares of the six infinitesimal operators with proper signs and the I is the sum of the products of dual infinitesimal operators $-I_{01}I_{23}-I_{02}I_{31}-I_{03}I_{12}$. However, if both 4κ and $4\kappa'$ are real and integers, two irreducible representations are associated with the pair, except in the case $\kappa=\kappa'=-\frac{1}{2}$.

If the representation is to be unitary the κ and κ' must have special forms. The results of this section can be compared directly with Bargmann's but the correspondence does not extend to all details [in particular, the reviewer believes that the $+n/2$ and $-n/2$ in (1a) should be replaced by $+n/4$ and $-n/4$]. However, the author's work clearly demonstrates that the well-known finite-dimensional representations of the homogeneous Lorentz group are only one class of representations which cannot be brought into unitary form.

In the remainder of the article, the author defines "expinors" which are generalizations of Dirac's "expansors" [cf. same Proc. **183**, 284–295 (1945); these Rev. **6**, 145] for the case of double-valued representations. Finally, invariant wave equations are given in which the components of the wave function are vectors in the space of the above representations. These equations may have physical significance. They are, of course, in general reducible in the sense that their solutions can be decomposed into linear subsets which are separately invariant under Lorentz transformations.

E. P. Wigner (Princeton, N. J.).

Bargmann, V. Irreducible unitary representations of the Lorentz group. Ann. of Math. (2) **48**, 568–640 (1947).

The (connected) Lorentz group $\mathfrak{L}_4$ and the group $\mathfrak{L}_3$ (the Lorentz group with two instead of three space coordinates) are known not to admit any nontrivial finite dimensional unitary representation. The author now classifies all (strongly continuous) irreducible representations of $\mathfrak{L}_3$ and $\mathfrak{L}_4$ (or rather of the spinor groups $\mathfrak{S}_3$ and $\mathfrak{S}_4$, which are double coverings of $\mathfrak{L}_3$ and $\mathfrak{L}_4$, thus including the double-valued representation of $\mathfrak{L}_3$ and $\mathfrak{L}_4$) by unitary operators in an infinite-dimensional Hilbert space $\mathfrak{H}$. The present paper treats $\mathfrak{S}_3$; the results for $\mathfrak{S}_4$ are briefly stated, and will be presented in detail in a second part. The basic idea is to consider the "infinitesimal representation": given a unitary representation of $\mathfrak{S}_3$, one associates with every element χ of the Lie algebra, by way of the corresponding 1-parameter subgroup and Stone's theorem, a self-adjoint operator H_χ, the "infinitesimal operator," with domain $\mathfrak{D}_\chi$ dense in $\mathfrak{H}$. [The assumptions which the author has to make concerning the $\mathfrak{D}_\chi$'s (p. 601, eq. 5.6) are unnecessary as has been shown by L. Gårding; cf. the following review.] Now a definite base χ_0, χ_1, χ_2 is chosen for the Lie algebra of $\mathfrak{S}_3$, with χ_0 generating a closed 1-parameter subgroup; H_0, H_1, H_2 are the corresponding operators; H_0 has a pure point spectrum. Irreducibility and Schur's lemma imply that the operator $Q=(H_0)^2-(H_1)^2-(H_2)^2$ (Casimir's operator) is a multiple $q\cdot I$ of the identity, q depending on the representation. Starting with a proper vector g of H_0, considering all the transforms of g under H_0, H_1, H_2 (these transforms span $\mathfrak{H}$ by irreducibility), and using the commutation rules, one finds the following cases, which exhaust all possibilities: either the spectrum of H_0 consists of the values k, $k+1$, $k+2, \cdots$ [$-k, -(k+1), -(k+2), \cdots$] for some positive integral or half integral k and then $q=k(1-k)$; these are the "discrete" classes D_k^+, D_k^-, respectively; or the spectrum of H_0 consists of all integers [half integers $\frac{1}{2}(2n-1)$], and then $q>0$[$q>\frac{1}{4}$]; these are the "continuous" classes C_q^0, $C_q^{\frac{1}{2}}$, respectively. Since a representation is determined by its infinitesimal representation, this amounts to a classification of all representations.

It is then shown by construction of explicit examples that all cases found actually occur. To this end the author introduces the concept of multiplier representation, generalizing for a group G, which acts on a space M, the customary regular representation in the vector space of functions $f(x)$ on M. He defines, for $a\in G$, the operator $T(a)$ by $T(a)f(x)=\mu(a, a^{-1}x)\cdot f(a^{-1}x)$, with a suitably chosen function $\mu(a, x)$, the "multiplier" of the representation. The spaces M considered are the unit circle and its interior, on which $\mathfrak{S}_3$ acts as group of projective, respectively conformal transformations. The author then discusses the "matrix elements," i.e., the inner products $(f, T(a)g)$, for fixed f and g, as functions of the group element a, for the various irreducible representations T. The equation $Q=q\cdot I$ for the Casimir operator means that the matrix elements satisfy a second order differential equation; they are given explicitly in terms of hypergeometric functions. Their asymptotic behavior, as a "goes to infinity," is discussed. For the discrete classes $D_k^{\pm}$, $k>\frac{1}{2}$, but not for $k=\frac{1}{2}$, the matrix elements are square integrable with respect to Haar measure. For the continuous classes with $q>\frac{1}{4}$ they can be made square integrable by integrating over the parameter s ($s^2=q-\frac{1}{4}$) with respect to any square integrable density. The operators obtained are denoted by $B_0(a)$ and $B_1(a)$. For $0<q<\frac{1}{4}$ this cannot be done. Orthogonality relations are found: $D_k^{\pm}$ and C_q^0, $0<q<\frac{1}{4}$, have to be left aside. Let f, g, f', g' be four vectors in the appropriate Hilbert space. Then

$$\int \overline{(f, A(a)g)}\cdot(f', A'(a)g')da=c\overline{(f, f')}\cdot(g, g'),$$

if A and A' are either the same representation in a discrete class $D_k^{\pm}$, $k>\frac{1}{2}$, or both B_0's, or both B_1's; c is a constant depending on A and A' only. In all other cases the integral is 0. Finally it is shown that the functions $(f, T(a)g)$, with arbitrary f and g and T running through $D_k^{\pm}$, $k>\frac{1}{2}$, all B_0's and B_1's, span the space of square integrable functions on $\mathfrak{S}_3$. Again $D_k^{\pm}$, and C_q^0, $q<\frac{1}{4}$, have to be left aside. A brief description of the results for $\mathfrak{L}_4$ is given; there are two quadratic operators Q and R which in every irreducible representation are multiples $q\cdot I$ and $r\cdot I$ of the identity. The representations then fall into the classes C_q^0 with $r=0$, $q>0$, and $C_{k,r}$ with r arbitrary real, k positive half integral or integral, and $q=1-k^2+(r/k)^2$. [The same results have been obtained by Gelfand and Neumark; cf. Acad. Sci. USSR. J. Phys. **10**, 93–94 (1946); these Rev. **8**, 132.]

H. Samelson (Ann Arbor, Mich.).

Gårding, Lars. Note on continuous representations of Lie groups. Proc. Nat. Acad. Sci. U. S. A. **33**, 331–332 (1947).

Let G be a Lie group, and let $T(a)$ ($a\in G$) be a strongly continuous representation of G by bounded operators on a Banach space B (with $T(e)=1$). For a one-parameter subgroup $A=a(s)$ of G define $T_A x=\lim_{s\to 0} s^{-1}(T(a(s))-1)x$, with $x\in B$; the x's for which the limit exists constitute the domain of T_A. The author proves, extending a result of Gelfand [C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 713–718 (1939); these Rev. **1**, 338], that the intersection of the domains of all the T_A's is dense in B; this result can be used to simplify the discussion in the work of V. Bargmann on the irreducible representations of the Lorentz group [see

the preceding review]. The method is to consider the subset B_r of B, containing the elements of the form $\int_G h(b)T(b)x db$, with $x \in B$, and $h(b)$ a function of class C_r on G, vanishing outside some compact set. Then every B_{r+1} is shown to be dense in B (by taking h close to the Dirac δ-function); it is in the domain of each T_A, and $T_A B_{r+1} \subset B_r$. The theorem is extended to semigroups and a still more general situation where the group operation of G is replaced by a mapping of $G \times G$ into G with certain group-like properties.

H. Samelson (Ann Arbor, Mich.).

Godement, Roger. Sur les relations d'orthogonalité de V. Bargmann. I. Résultats préliminaires. C. R. Acad. Sci. Paris **225**, 521–523 (1947).

Godement, Roger. Sur les relations d'orthogonalité de V. Bargmann. II. Démonstration générale. C. R. Acad. Sci. Paris **225**, 657–659 (1947).

The author announces results extending the orthogonality relations for unitary representations of compact groups and certain orthogonality relations due to V. Bargmann [cf. the second preceding review]. Bargmann showed that for the projective group on the line, which is noncompact, there exist irreducible continuous unitary representations $a \to U(a)$ of the group on a Hilbert space H such that, for any x and y in H, $(U(a)x, y)$ is square integrable over the group, relative to Haar measure, and that the analogue of the orthogonality relations for compact groups is valid for such representations. The author outlines a proof that for any locally compact group, such representations, when they exist, satisfy the orthogonality relations. The main tools are Schur's lemma for representations on Hilbert space and previous work of the author concerning square integrable positive definite functions on groups. *I. E. Segal* (Chicago, Ill.).

Godement, Roger. Analyse harmonique dans les groupes centraux. II. Formule d'inversion de Fourier. C. R. Acad. Sci. Paris **225**, 221–223 (1947).

The author announces extensions of the results of a previous note [same C. R. **225**, 19–21 (1947); these Rev. **9**, 8], consisting of the following assertions about central groups [for terminology, see the review of part I]. Every irreducible continuous unitary representation is finite-dimensional. There is a natural one-to-one correspondence between such representations and the characters. The analogue of the Fourier inversion formula is valid for a function on the group which is both integrable and a linear combination of positive definite functions. The author points out that the first assertion implies that a connected central group is the direct product of a compact and an Abelian group.

I. E. Segal (Chicago, Ill.).

Baer, Reinhold. Direct decompositions. Trans. Amer. Math. Soc. **62**, 62–98 (1947).

The present paper contains a broadly based theory of direct decompositions of operator loops. A loop L admitting a certain system M of operators is termed an M-loop L. If η is an M-endomorphism, an M-subloop C of L is termed a complement of η if every coset of $L/R(\eta)$ contains one and only one element in C and if $C = C\eta$; here $R(\eta)$ stands for the radical of η. The M-loop L is the direct sum of A and B if A and B are normal M-subloops of L whose cross-cut is 0 and whose sum is L. This notion extends in the usual way for several summands. One of the basic notions in the author's theory is that of complementary decomposition endomorphisms, which means M-endomorphisms α, β satisfying the following conditions: $\alpha^2 = \alpha$, $\beta^2 = \beta$, $\alpha\beta = \beta\alpha = 0$, $\alpha + \beta = \beta + \alpha = 1$. A member of a pair of complementary decomposition endomorphisms is called a decomposition endomorphism. The M-loops whose direct decompositions are studied are required to verify the following splitting hypothesis: If δ, ϵ is a pair of complementary decomposition endomorphisms of the M-loop L, and if α is a decomposition endomorphism of L, then the endomorphism $\alpha\delta\alpha + \alpha$ of L possesses one and only one complement. The author proves first a special refinement theorem for decompositions into two direct summands and from this he deduces the general refinement theorem asserting that any two direct decompositions of L possess exchange isomorphic refinements. Further results concern groups without Abelian direct summands; direct decompositions into summands that are either Abelian and indecomposable or else do not possess Abelian direct summands, not 0; and finally applications of the splitting criteria to the general refinement theorem. The author's investigations are of great generality as they include both Schmidt's and Kořínek's decomposition theorems of which neither is a consequence of the other.

O. Borůvka (Brno).

Liapin, E. The kernels of homomorphisms of associative systems. Rec. Math. [Mat. Sbornik] N.S. **20**(62), 497–515 (1947). (Russian. English summary)

The author considers associative systems S which satisfy all the usual group axioms except for the existence of an inverse. A subsystem N of S is called normal if, for $a, b \in S$ and $n \in N$, the elements anb and ab both belong or both do not belong to N. Homomorphisms $S \to S'$ of such systems are considered. The usual connection between the concept of the kernel of a homomorphism and that of a normal subgroup is explored in detail. *S. Eilenberg.*

NUMBER THEORY

***Ferrier, A. Les Nombres Premiers. Principaux résultats obtenus depuis Euclide. Table donnant, jusqu'à 100.000, les nombres premiers et les nombres composés n'ayant pas de diviseur inférieur à 17, avec, pour chacun d'eux, son plus petit diviseur.** Librairie Vuibert, Paris, 1947. vi+111 pp. 280 francs.

The last half [pp. 60–110] of this book contains the factor table so completely described in the title. This compact little table will be found useful in problems involving isolated 5-digit numbers. Primes are indicated by bold face type. [Erratum: The final digits of the prime 99989 should be in bold face.] No indication of the method used in the construction of the table or of its comparison with any other such table is indicated.

The first half of the book gives in eleven short chapters an interesting summary of classical results concerning primes, mostly without proof. *D. H. Lehmer.*

Jarden, Dov. Addenda to the table of Fibonacci numbers. Riveon Lematematika **2**, 22 (1947). (Hebrew)

This list supplements a previous table [same Riveon **1**, 35–37 (1946); these Rev. **8**, 135] of the factors of Fibonacci's U_n and $V_n = U_{2n}/U_n$. Most of these addenda have resulted from a comparison with the table of M. Kraïtchik [Re-

cherches sur la Théorie des Nombres, v. 1, Paris, 1924, pp. 77–80]. Fourteen hitherto unpublished factorisations are given. *D. H. Lehmer* (Berkeley, Calif.).

Bambah, R. P., Chowla, S., Gupta, H., and Lahiri, D. B. Congruence properties of Ramanujan's function $\tau(n)$. Quart. J. Math., Oxford Ser. **18**, 143–146 (1947).

Another proof is given of the fact that, if d is any divisor of 24 which is prime to n, then $\tau(n)\equiv\sigma(n) \pmod d$. [Cf. Gupta, J. Indian Math. Soc. (N.S.) **9**, 59–60 (1945); Bambah, Chowla and Gupta, Bull. Amer. Math. Soc. **53**, 766–767 (1947); Bambah and Chowla, ibid., 768–769 (1947); these Rev. **8**, 10; **9**, 79.] *D. H. Lehmer.*

Segre, B. On arithmetical properties of quartic surfaces. Proc. London Math. Soc. (2) **49**, 353–395 (1947).

Let $F(x, y, z, w)$ be a quartic form with rational coefficients. Certain phases of the general problem of finding rational solutions of $F=0$ are handled by geometric methods. The author restricts himself to the case in which the quartic surface F contains one or more rational lines and considers the possibility of finding other one-, or two-, parameter families of rational solutions. His main tools are two transformations defined as follows. Let C be a rational line of F and P a rational point of C. The tangent plane to F at P intersects F in C and a residual cubic Γ which also contains P. The tangent line to Γ at P intersects Γ again in a rational point P', generally distinct from P. The locus of P' as P describes C is a curve C', called the R-transform of C, which contains a one-parameter set of rational points of F. If C and D are two skew rational lines of F, to a general point P of F there corresponds a unique point P' of F such that the line PP' intersects both C and D. Then P' is the T-transform of P, and the T-transform of any set of rational points is a set of rational points. Using these transformations and other geometric properties of quartic surfaces the author investigates surfaces containing various configurations of lines; special attention is also given to surfaces of the form $\phi(x, y)=\psi(z, w)$. As a particularly interesting case the equation $x^4+y^4=z^4+w^4$ is shown to have an infinite number of one-parameter sets of rational points in addition to the one such set discovered by Euler. [Dickson, History of the Theory of Numbers, v. 2, Washington, 1920, p. 644.] *R. J. Walker* (Ithaca, N. Y.).

Stoll, A. Das Proportionalwahl-Problem als diophantische Näherungsaufgabe. Vierteljschr. Naturforsch. Ges. Zürich **92**, 204–212 (1947).

With elections under a many party system one faces the problem of a fair apportionment of seats to the various parties [or states in the U. S. House of Representatives]. Let p_k be the fraction of votes cast for the kth party [population in the kth state]. The problem consists in finding integers n_k such that $\sum n_k=n$ and that the system $\{n_k\}$ comes as near as possible to the system $\{np_k\}$. The deviation of n_k from np_k is measured by a convex function $f_k(n_k)$ and the total deviation of $\{n_k\}$ from $\{np_k\}$ is defined as the sum of the individual deviations. Typical examples are $f_k(n_k)=|n_k-np_k|$, $(n_k-np_k)^2$ and $(n_k-np_k)^2/np_k$. The optimal solution is found for these and other cases. *W. Feller* (Ithaca, N. Y.).

***Drewes, Aart. Diophantische Benaderingsproblemen. [Diophantine Approximation Problems].** Thesis, Free University of Amsterdam, 1945. vi+72 pp.

Chapter I contains a general survey on the metrical theories of Diophantine approximation and a summary of the author's results. Chapter II gives a generalization for the many-dimensional case of a theorem of the reviewer [Nederl. Akad. Wetensch., Proc. **39**, 225–240 (1936)]. The method used is a transcendental one and is based on a theorem of van der Corput and Koksma [see the reviewer's book, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, Satz 4]. Chapter III uses the same method and a theorem of Vinogradov and deals with the uniform distribution of systems of polynomials $\{f_\nu(x)\}$ $(\nu=1, \cdots, n; x=1, 2, \cdots)$. Chapters IV to VI deal with the distribution (mod 1) of slowly increasing functions, e.g., $(\theta x)^{1/k}$ $(k>1)$ and chapter V contains similar research on functions $f(x)$ whose nth differential quotient is a slowly increasing function. This chapter is remarkable for the elementary method used, although the results are less sharp than those developed by other authors with use of transcendental methods. Most of the theorems proved by the author are very general; in most cases an upper bound for the remainder in the formula of uniform distribution is given. It would take too much space to quote particular results. *J. F. Koksma* (Amsterdam).

Popken, J. On the irrationality of the tangent of a rational number. Norsk Mat. Tidsskr. **26**, 66–70 (1944). (Dutch)

The author proves the following theorem: If r designates a rational number, not 0, and ϵ an arbitrarily chosen positive constant, then we have for nearly all pairs of integers q, p (q positive) the inequality $|\tan r-p/q|>q^{-2-\epsilon}$. [For the method used see Popken, Nederl. Akad. Wetensch., Proc. **43**, 712–714 (1940); these Rev. **2**, 149.] The author does not make use of continued fractions, in distinction to Lambert in his classic investigations. *J. F. Koksma.*

Venkov, B. A. Sur le problème extrémale de Markoff pour les formes quadratiques ternaires indéfinies. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 429–494 (1945). (Russian. French summary)

Consider an indefinite ternary quadratic form $f(x_1x_2x_3)=x'Ax$, $A=(a_{ij})$, $x'=[x_1x_2x_3]$, with real coefficients a_{ij} and of determinant $|A|=1$. The minimum $m(f)$ of a form f is the numerically smallest value attained by $f(x_1x_2x_3)$ for integral values of $x_1x_2x_3$ not all zero. The author constructs eleven forms $f_1, \cdots, f_{11}$ of determinant unity whose minima are equal, respectively, to $(\frac{2}{5})^{\frac{1}{2}}$, $(\frac{2}{5})^{\frac{1}{2}}$, $(\frac{1}{3})^{\frac{1}{2}}$, $(\frac{8}{25})^{\frac{1}{2}}$, $(\frac{79}{1200})^{\frac{1}{2}}$, $(\frac{2}{7})^{\frac{1}{2}}$, $(\frac{4}{15})^{\frac{1}{2}}$, $(\frac{59}{3\cdot 13^2})^{\frac{1}{2}}$, $(\frac{39}{112})^{\frac{1}{2}}$, $(\frac{29}{135})^{\frac{1}{2}}$, $(\frac{2}{9})^{\frac{1}{2}}$, in absolute value, and proves that unless a real indefinite ternary quadratic form f of determinant 1 is equivalent to one of the above eleven forms, we have $|m(f)|<(\frac{2}{9})^{\frac{1}{2}}$. Employing similar information regarding the minima of indefinite binary quadratic forms, Markoff [Math. Ann. **56**, 233–251 (1902); Mém. Acad. Imp. Sci. St. Pétersbourg (8) **23**, no. 7 (1909)] determined the first four minima. By similar methods Dickson [Studies in the Theory of Numbers, University of Chicago Press, 1930] recomputed these four minima. Fujiwara [Abh. Math. Sem. Hamburgischen Univ. **2**, 74–80 (1923)] used difficult geometric considerations to arrive at an incorrect value of $m(f_4)$. The above sequence of eleven minima and the sequence of associated forms represents the first significant extension of the results of Markoff. The method

employed by the author is an extension of his geometrical ideas employed in the study of the integral automorphs of the indefinite ternary quadratic forms [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 1, 139–170 (1937)]. It makes use of information regarding the distribution with respect to the cone $f(x, y, z)=0$ of a set of nonequivalent (under the automorphisms of f) lattice points.

A. E. Ross (Notre Dame, Ind.).

Eichler, Martin. Über gewisse Anzahlformeln in der Theorie der quadratischen Formen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1943**, 1–24 (1944).

Consider a set of integral quadratic forms in $2n$ variables, all of the same determinant, which can be carried into one another by a rational linear transformation and division by a rational number. Let $f_i=x'A_ix$ be a set of representatives of the different classes in this set and let T_{ik} be an integral matrix such that $T'_{ik}A_iT_{ik}=tA_k$. As in a later paper [Festschrift zum 60. Geburtstag von A. Speiser, pp. 34–46, Zürich, 1945; these Rev. 7, 369] the author calls such a matrix a transformer of norm t belonging to A_i and A_k and discusses a correspondence between these transformers and ideals of a simple algebra. He studies the factorization properties of these transformers, establishes a relation between the number of transformers of a given norm and the number of representations of integers by quadratic forms, and obtains in a purely number-theoretic manner equations corresponding to the representation formulae first obtained by analytic methods by Hecke [Lectures on Dirichlet Series, Institute for Advanced Study, Princeton, N. J., 1938].

A. E. Ross (Notre Dame, Ind.).

Ren'i, A. A. On the representation of an even number as the sum of a single prime and a single almost-prime number. Doklady Akad. Nauk SSSR (N.S.) **56**, 455–458 (1947). (Russian)

A set of integers S, with the property that there exists an absolute constant K such that each $x\varepsilon S$ has at most K distinct prime factors, is called an almost-prime set. Each $x\varepsilon S$ is called an almost-prime number. The author indicates the proof, to be given in detail elsewhere, that each even integer is the sum of an almost-prime number (taken from a fixed set S) and a prime number. He also states that he can prove that there exist infinitely many primes p such that $p+2$ is almost-prime (being in a fixed set S^*).

The first result, regarding the representation of an even number, is an approximation to the unproved Goldbach conjecture and supersedes an earlier proof of the same proposition by Estermann [J. Reine Angew. Math. **168**, 106–116 (1932)] which made use of an unproved generalized Riemann hypothesis for all Dirichlet L-series. The second result is an approximation to the conjecture of the existence of infinitely many twin primes and is apparently a new result.

In order to formulate the basic result enabling the author to dispense with the Riemann hypothesis, recall that if $(p, q)=1$ and $D=pq$ then any character $\chi_D(n)$ modulo D can be uniquely decomposed into the product $\chi_p(n)\chi_q(n)$, where the new characters are to the moduli p and q, respectively. The author calls $\chi_D(n)$ primitive relative to p if $\chi_p(n)$ is not the principal character modulo p. The author's result, for which no proof is indicated, is the following. Let q be a square-free integer and $c_1>0$ an absolute constant. Then there exists a constant $\delta>0$ such that, if $A\geqq c_1$, $k=(\log q)/(\log A)+1\leqq\log^3 A$, p is any prime such that $(p, q)=1$ and $A\leqq p\leqq 2A$ (there being, asymptotically for large A, $A/\{\varphi(q)\log A\}$ such p) with the possible exception of $A^{3/4}$ values, and if $\chi(n)$ is any character mod pq which is primitive relative to p, then $L(\sigma+it, \chi)=\sum_{n=1}^{\infty}\chi(n)n^{-\sigma-it}$ has no zeros in the rectangle $1-\delta/(k+1)\leqq\sigma\leqq 1$, $0\leqq|t|\leqq\log^3 pq$.

To prove his result on the Goldbach conjecture, the author considers $H(2N)=\sum\log p\cdot\exp\{-p(\log 2N)/(2N)\}$, extended over those primes $p<2N$ such that $(2N-p, B)=1$; here $B=\prod p^*$, extended over those primes p^* such that $c_2\leqq p^*\leqq(2N)^{1/R}$, where R is a suitably chosen integer. It is clear that $P=2N-p$ is almost-prime and hence, for the weak Goldbach theorem, it is sufficient to prove that there exists a $c_7>0$ such that if $N\geqq c_7$ then $H(2N)>0$; for then we have that there exists an almost-prime P and a prime p such that $2N=P+p$. The author states that an application of Brun's sieve method [Skrifter Videnskapsselskapets i Kristiania. I. Mat.-Nat. Kl. **1920**, no. 3] gives

$$H(2N)>c_3N/\log^2 N-\sum_{Q\in E}|R_Q(2N)|,$$

where E is a certain set and

$$R_Q(x)=\sum\log p\cdot\exp(-p\log x/x)-x/\{\varphi(Q)\log x\},$$

where the sum extends over all the primes $p<x$ such that $p\equiv l \bmod Q$, where $(l, Q)=1$. Using the above result on $L(s, \chi)$ and the results of Titchmarsh [Rend. Circ. Mat. Palermo **54**, 414–429 (1930); **57**, 478–479 (1933)], Page [Proc. London Math. Soc. (2) **39**, 116–141 (1935)] and Siegel [Acta Arith. **1**, 83–86 (1935)], the author indicates a proof of the estimate

$$\sum_{Q\in E}|R_Q(2N)|<4N/\log^3 N$$

from which it follows that $H(2N)>0$ for $N\geqq c_7$.

L. Schoenfeld (Cambridge, Mass.).

ANALYSIS

Natanson, I. P. On an inequality. Doklady Akad. Nauk SSSR (N.S.) **56**, 911–913 (1947). (Russian)

Let $f(x)$ be integrable in $[a, b]$, $F(x)=\int_a^x f(t)dt$, $|F(x)|\leqq M(x-a)$ for $a<x\leqq b$; $g(x)$ nonnegative, nonincreasing and integrable; the author proves the inequality

$$(*)\qquad\left|\int_{a+}^{b}f(x)g(x)dx\right|\leqq M\int_a^b g(x)dx,$$

which he states is of use in studying the representation of functions by singular integrals. His proof (slightly modified) is as follows. Let $a<\alpha<\beta\leqq b$; we have $\int_a^\alpha g(x)dx\geqq(\alpha-a)g(\alpha)$. Then

$$\left|\int_a^\beta f(x)g(x)dx-g(\beta)\int_a^\beta f(x)dx\right|=\left|\int_a^\beta f(x)[g(x)-g(\beta)]dx\right|$$

$$\leqq\left|F(\alpha)[g(\alpha)-g(\beta)]\right|+\left|\int_\alpha^\beta F(x)d[-g(x)+g(\beta)]\right|$$

$$\leqq M(\alpha-a)g(\alpha)+M\int_\alpha^\beta(x-a)d[-g(x)+g(\beta)]$$

$$\leqq 2M\int_a^\alpha g(x)dx+M\int_\alpha^\beta g(x)dx-Mg(\beta)(\beta-\alpha).$$

Letting α and β approach a, we see that $\int_{a+}^{b} f(x)g(x)dx$ exists. Taking $\beta=b$ and letting $\alpha\to a$, we obtain

$$\left|\int_{a+}^{b} f(x)g(x)dx\right| \leqq M\int_{a}^{b} g(x)dx+g(b)[F(b)-M(b-a)],$$

which is a little stronger than (*). *R. P. Boas, Jr.*

Ghizzetti, A. Condizioni necessarie e sufficienti per i momenti di una funzione limitata. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 533–536 (1947).

The author studies the problem of existence in a bounded interval (a, b) of a quasicontinuous $f(x)$ with $0\leqq f(x)\leqq 1$ and prescribed moments μ_k $(k=0, 1, \cdots)$. Define σ_k and δ_n formally by $1-\sum\sigma_k z^{k+1}=\exp(-\sum\mu_k z^{k+1})$ and $1+\sum\delta_k z^{k+1}=\exp(\sum\mu_k z^{k+1})$. The Hankel determinants with first row $(\sigma_0, \cdots, \sigma_k)$ and $(\delta_0, \cdots, \delta_k)$ are equal and denoted by Δ_k. Let Δ_k' be the determinant whose first row is $(1, a, a^2, \cdots, a^k)$ and the $(j+1)$th row $(\sigma_{j-1}, \sigma_j, \cdots, \sigma_{j+k-1})$. Let $(-1)^k\Delta_k''$ be the determinant obtained on replacing a by b and σ_j by δ_j. A necessary and sufficient condition for the existence of a solution $f(x)$ is that either (1) all Δ_k, Δ_k', Δ_k'' are positive, or (2) all vanish, or (3) that for some n we have $\Delta_j>0$ for $0\leqq j\leqq n-1$, and $\Delta_j=0$ for $j\geqq n$ and at the same time $\Delta_j'>0$, $\Delta_j''>0$ for $1\leqq j\leqq n-1$, $\Delta_n'\geqq 0$, $\Delta_n''\geqq 0$, and $\Delta_j'=\Delta_j''=0$ for $j>n$. The necessity of the condition was proved previously [Atti Accad. Italia Mem. Cl. Sci. Fis. Mat. Nat. (7) **13**, 1165–1199 (1942); these Rev. **6**, 128]. *W. Feller* (Ithaca, N. Y.).

Theory of Sets, Theory of Functions of Real Variables

Novikov, P. S. The power of the set of connected components of an A-set. Doklady Akad. Nauk SSSR (N.S.) **56**, 787–790 (1947). (Russian)

The following theorem is proved. Let E be any analytic set in n-dimensional Euclidean space. Then the cardinal number of the family of all connected components of E can be only a finite number, $\aleph_0$, or $2^{\aleph_0}$. The problem of determining possible cardinal numbers for the family of connected components consisting of exactly one point is mentioned and is left open. *E. Hewitt* (Chicago, Ill.).

Rado, R. A theorem on general measure. J. London Math. Soc. **21** (1946), 291–300 (1947).

Suppose M is a field of subsets of Euclidean n-space R; every open halfspace is an element of M; φ is a finite nonnegative function on M; $\varphi(A)\leqq\varphi(B)+\varphi(C)$ whenever $A, B, C\epsilon M$ and $A\subset(B\cup C)$; for each $\epsilon>0$ there is a bounded set $A\epsilon M$ for which $\varphi(R-A)<\epsilon$. Then there is a point $x\epsilon R$ such that $\varphi(A)\geqq(n+1)^{-1}\varphi(R)$ whenever A is an open halfspace for which $x\epsilon A$. *H. Federer* (Providence, R. I.).

Halmos, Paul R. Invariant measures. Ann. of Math. (2) **48**, 735–754 (1947).

The general problem discussed in this paper is the existence of a measure invariant with respect to a given transformation T and suitably related to a given measure m. The given measure m is defined on a Borel field B of subsets of a space X and the given transformation T is a one-to-one transformation of X onto itself such that both T and its inverse take sets of B into sets of B. A measure μ defined on B is finite if $\mu(X)<\infty$ and σ-finite if X is the countable sum of sets of B having finite measure. A measure μ is stronger than m if $m(A)=0$ whenever $\mu(A)=0$, and μ is equivalent to m if it is both stronger and weaker than m. A set A is ∞-equivalent to a set B if A and B have disjoint decompositions $A=\cup_1^\infty A_i$, $B=\cup_1^\infty B_i$ such that, for some $j(i)$, $B_i=T^{j(i)}A_i$. A subset B of A is a proper subset if $m(A-B)>0$. A set A is compressible if $T(A)$ is a proper subset of A. A set A is bounded if there is no proper subset of A to which A is ∞-equivalent. The pair (m, T) is nonsingular if $m(A)=0$ implies $m(T^iA)=0$ for every i.

The paper centers around the following results. (I) The nonsingularity conditions can be removed from the theorem of E. Hopf [Trans. Amer. Math. Soc. **34**, 373–393 (1932)] that there is a finite measure invariant under T and equivalent to m if and only if X is bounded. (II) An example is given of an incompressible and nonsingular T which preserves no finite measure either weaker or stronger than m. (III) It is shown that there exists a σ-finite invariant measure stronger than m if and only if X is a countable sum of bounded sets. (IV) The above results leave unsolved the question as to the existence of a T having no invariant σ-finite measure. The author rephrases this problem in several ways and offers as a possible tool for its solution the following structure theorem: every one-to-one measurable and nonsingular transformation T of the unit interval onto itself is a product $T=PS$, where P is measure-preserving and S is simple, in the sense that there are disjoint sets A and B such that S is the identity on A and the iterated images of B under S are all disjoint and add up to the complement of A. The paper is provided throughout with an ample supply of examples and helpful remarks.

L. H. Loomis (Cambridge, Mass.).

Edelstein, Michael. On a set characterizing relative translations of 2 point-sets. Riveon Lematematika **1**, 91–94 (1947). (Hebrew)

Hartman, P. On the limits of Riemann approximating sums. Quart. J. Math., Oxford Ser. **18**, 124–127 (1947).

It is known that the closure of the set of limits (as the norm of the subdivision approaches zero) of Riemann sums of a bounded real function of a bounded real variable is a closed interval. Upon replacing "real function" by "k-dimensional vector function" it is shown that "interval" may be replaced by "convex set." Moreover $\epsilon>0$ implies a $\delta>0$ such that, for a subdivision with norm less than δ and P in the convex set, there is a Riemann sum over this subdivision differing from P by less than ϵ. *J. F. Randolph.*

Radó, Tibor. The isoperimetric inequality and the Lebesgue definition of surface area. Trans. Amer. Math. Soc. **61**, 530–555 (1947).

Let f be a map of the oriented 2-sphere S^2 into 3-space E^3. For each $p\epsilon(E^3-f(S^2))$, the 2-dimensional singular homology group of $(E^3-\{p\})$ with integer coefficients is infinite cyclic, and the pair (S^2, f) represents an integral multiple of a generator of this group. Let $i(p)$ be the absolute value of the integral multiplier. Let V be the Lebesgue integral of i with respect to 3-dimensional measure and let A be the Lebesgue area of f. Then the isoperimetric inequality $36\pi V^2\leqq A^3$ holds. The proof proceeds by passage to the limit from the simplicial case. In discussing the simplicial case, the author considers arbitrary finite sets of triangles in 3-space and integral valued functions (somewhat analogous to i) which are constant on each component of the complement of the union of the triangles, and jump at most

by one upon crossing any triangle of the set. The classical isoperimetric inequality for surfaces of revolution is assumed. *H. Federer* (Providence, R. I.).

Reichelderfer, P. V. On the definition of the essential multiplicity for continuous transformations in the plane. Trans. Amer. Math. Soc. 62, 284–314 (1947).

If T is a bounded map of the bounded open connected subset D of the plane into the plane, z is a point of the range of T and $\epsilon>0$, then there is a bounded map T' of D into the plane whose distance (defined as a supremum) from T is less than ϵ, and such that the number of points w in D for which $T'(w)=z$ equals the number of essential components of the T counterimage of z. *H. Federer.*

Reichelderfer, Paul V. The essential part of a surface. Bull. Amer. Math. Soc. 53, 845–855 (1947).

Expository lecture. *H. Federer* (Providence, R. I.).

Stampacchia, G. Sulla definizione assiomatica dell'area di una superficie rettificabile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 2, 542–546 (1947).

If φ is any function on the class of all rectifiable surfaces (that is, all functions which map some plane region bounded by rectifiable Jordan curves into Euclidean three-space and satisfy a Lipschitz condition) which agrees with the elementary area for polyhedra, is lower semicontinuous, and is additive whenever the domain is cut by an arc such that the corresponding curve on the surface has finite length, then φ agrees with Lebesgue area for all rectifiable surfaces. *H. Federer* (Providence, R. I.).

Theory of Functions of Complex Variables

***Ritt, J. F. Theory of Functions.** King's Crown Press, New York, 1947. x+181 pp. $2.50.

This is a "basic" introduction into the theory of functions: a coherent series of short and to the point lecture notes giving the student an account of the fundamental definitions and theorems of the subject. The proofs are both reliable and very readable. Little detail and only a few examples are added. But students and lecturers alike will welcome these clear and clean notes as a handy précis of a more elaborate presentation. The emphasis is laid on the theory of the analytic functions of a complex variable. However, the notes start with the definition and properties of the real numbers (as decimals). Limits, series, sets of points, differentiation, and Riemann's integral indicate a few further stages in real analysis. In the complex field, both Cauchy's theory of curvilinear integrals and Weierstrass's theory of power series are treated in their essentials. Topological requisites, like Jordan's theorem, are plainly stated. Weierstrass's products for integral functions and Mittag-Leffler's sums for meromorphic functions are treated, and the notes end with the notion of the complete analytic function.

The author's attitude is strictly "formal." Thus his decimals, even when finite, are "meaningless symbols" of digits which become mathematical entities (real numbers) only through the rules defining a "game" with them. Though the reviewer shares the formal view, he thinks that the author cannot claim (as he does in the preface) that his treatment is less abstract than that of Cantor or Dedekind: the student is familiar with significant decimals but not with symbolic ones. By inviting him to think of the latter as if they were the former, the whole issue is confused. Fortunately, such invitation is made only in the preface, not in the text. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Perry, A. D., and Youngs, J. W. T. Remarks on analyticity and integration. Amer. Math. Monthly 54, 313–318 (1947).

A curve in the complex plane is taken as any continuous image of the segment $[0, 1]$, and a sequence of curves C_n converges to a curve C_0 if parametric representations of the C_n converge uniformly to one for C_0. Let $f(z)$ be a complex-valued continuous function in a region R, and if C is rectifiable let $I(C)$ be the integral of f along C. The authors show that the following four conditions are equivalent: (1) $f(z)$ is analytic in R; (2) $I(C_n)$ converges whenever C_n converges to a (not necessarily rectifiable) curve C_0 in R; (3) $I(C_n)\to 0$ whenever C_n converges to a point in R; (4) $\{|I(C_n)|\}$ is bounded whenever C_n converges to a point in R. *L. H. Loomis* (Cambridge, Mass.).

Bermant, A. On certain generalizations of E. Lindelöf's principle and their applications. Rec. Math. [Mat. Sbornik] N.S. 20(62), 55–112 (1947). (Russian. English summary)

Let $w(z)$, $w(0)=0$, be regular for $|z|<1$ and write $L_r(w)=(2\pi)^{-1}\int \log|w(z)|\,d\arg w(z)$, where the integral is extended in the positive sense over the image of $|z|=r<1$ by $w(z)$. The corresponding integral extended over the boundary of the star of $w(z)$ in the circle $|z|\leqq r$ will be denoted by $L_r^*(w)$. The quantities $L_r(w/r)$ and $L_r^*(w/r)$ are called the logarithmic and the logarithmic star distortion of $w(z)$ in $|z|\leqq r$. The author extends the Schwarz lemma and the more general principle of Lindelöf (which reduces to the Schwarz lemma in the case of simply-connected domains), replacing the condition that "the maximum modulus is bounded by unity" by the condition that "the logarithmic or logarithmic star distortion is nonpositive." For example: suppose that $w(z)/z\neq 0$ in $|z|<1$, let $\mathfrak{w}(z)$, $\mathfrak{w}(0)=0$, be a function which is regular in a domain D containing $z=0$, and let R, $\bar{R}$ be the Riemann surfaces generated respectively by $w(z)$ in $|z|<1$ and $\mathfrak{w}(z)$ in D. If R is subordinate to $\bar{R}$ and if the logarithmic distortion of the function $\xi(z)=\mathfrak{w}^{-1}[w(z)]$ is nonpositive, then $|w'(0)|\leqq|\mathfrak{w}'(0)|$ with equality only for $w(z)=\mathfrak{w}(\epsilon z)$, $|\epsilon|=1$, and the domain D coinciding with the circle $|z|<1$. This result remains valid even when $\bar{R}$ cannot be mapped on $|z|<1$ and so is more general than the principle of Lindelöf. In other formulations a nonpositive logarithmic star-distortion is assumed. Applications are given. *D. C. Spencer.*

***Heinhold, J. Zur praxis der konformen Abbildung.** Ber. Math.-Tagung Tübingen 1946, pp. 75–77 (1947).

If B is a simply-connected domain interior to the unit circle $|z|<1$ which contains $z=0$, the conformal mapping of B on the circle may be approximated by a sequence of functions $\varphi_1(z)=z_1$, $\varphi_2(z_1)=z_2$, $\cdots$, $\varphi_\nu(z_{\nu-1})=z_\nu$, ($\varphi_\nu(0)=0$, $\varphi_\nu'(0)>0$) which map B on domains $B_1, \cdots, B_\nu$ such that the minimum distance of the boundaries of these domains from the origin tends monotonically to unity. Ringleb [Habilitationsschrift, Heidelberg, 1939] has given a graphical procedure for carrying out this approximation using for φ_ν the well-known functions introduced by Koebe. The author suggests a modified procedure. The domain B is assumed to lie in the exterior of the unit circle and is to be mapped on $|z|>1$. Let the maximum distance of the

boundary of B from the origin be $1+\delta$, $\delta>0$, and suppose that $z=1+\delta$ is a boundary point and that the segment $(1, 1+\delta)$ lies outside B. The author advocates use of the function $\varphi=w+(w^2-1)^{\frac{1}{2}}$, where $w=(z+z^{-1}-d)/(2+d)$, $d=\frac{1}{2}\delta^2/(1+\delta)$, which carries $z=1+\delta$ into the point 1 and brings all other points of the boundary nearer to the unit circumference. *D. C. Spencer.*

Le-Van-Thiem, Beitrag zum Typenproblem der Riemannschen Flächen. Comment. Math. Helv. **20**, 270–287 (1947).

Let W_q be the class of open simply-connected Riemann surfaces covering the sphere whose branch points lie over the q points $a_1, \cdots, a_q$. The author considers the subclass of surfaces w_q of W_q satisfying: (a) the surfaces have only branch points of zero or infinite order; (b) they have no logarithmic ends; (c) at each vertex of the complex there are either two branches or exactly q branches issuing from it. The complex of any surface w_q (apart from vertices where two branches meet) is topologically equivalent to the complex of the universal covering surface w_q^0 of the sphere punctured at q points. If $q>2$ (as is supposed) the latter surface may be mapped onto the interior of the unit circle, the normal polygons G^n being bounded by q circular arcs orthogonal to the unit circle. Starting from a polygon G^0, there are q neighboring polygons G^1 (first generation) obtained by reflection on the sides of G^0, $q(q-1)$ polygons G^2 (second generation) corresponding to the free sides of polygons G^1, and in general there are $q(q-1)^{k-1}$ polygons G^k of the kth generation. Let s_{ki} denote the area of the polygon (ki), the ith polygon of the kth generation. To a polygon (ki) there corresponds a vertex of the complex of w_q^0 and therefore a vertex (with more than two branches) of the complex of w_q; the latter vertex will be denoted by (ki). One and only one vertex $(k-1, i')$ is connected with (ki) and it is connected by a piece of the complex consisting of alternating simple and $(q-1)$-ple lines whose number will be denoted by l_{ki} (the "length" of the piece). Let $l_k=\max l_{ki}$ $(i=1, 2, \cdots)$, $L_k=\max l_h$ $(h\leqq k)$. The author proves: (i) convergence of either of the series (a) $\sum_k\sum_i l_{ki}s_{ki}$, (b) $\sum_k l_k(q-1)^{-k}$ implies that the surface is of hyperbolic type; (ii) if l_{ki} is independent of i and monotonic increasing in k, then the convergence of the series $\sum_n 1/\sigma(n)$ is necessary and sufficient in order that the surface should be of hyperbolic type. Here $\sigma(n)$ is the number of boundary vertices of the subcomplex formed by the first n generations of the graph of w_q, and the necessity of the convergence follows from the Nevanlinna-Wittich criterion. The convergence of the series (b) in (i) is a sharper criterion than Kobayashi's. *D. C. Spencer* (Stanford University, Calif.).

Dugué, Daniel. Le défaut au sens de M. Nevanlinna dépend de l'origine choisie. C. R. Acad. Sci. Paris **225**, 555–556 (1947).

It is proved that the functions

$$\varphi(z)=(\exp(2\pi ie^z)-1)/(\exp(2\pi ie^{-z})-1)$$

and $\varphi(z+h)$, where $\Re h\neq 0$, have different defects in the sense of Nevanlinna. For a comment, see the following review. *L. Ahlfors* (Cambridge, Mass.).

Valiron, Georges. Valeurs exceptionnelles et valeurs déficientes des fonctions méromorphes. C. R. Acad. Sci. Paris **225**, 556–558 (1947).

Commenting on the preceding paper Valiron notes that the characteristic function $T(r, f)$ remains asymptotically invariant under a change of origin if

$$\lim_{r\to\infty} T(r+1, f)/T(r, f)=1.$$

In particular, this will be so if the upper and lower order of the function differ by less than 1. If this is the case, the defects are also invariant. It is also remarked that exceptional values in the sense of Borel are not necessarily defective in the sense of Nevanlinna. Such occurrences are of course always due to irregularities in the growth of $T(r, f)$. The author has suggested a revised definition of the defects which does not have these drawbacks and is such that the sum of defects still does not exceed 2. *L. Ahlfors* (Cambridge, Mass.).

Zorn, Max A. Note on power series. Bull. Amer. Math. Soc. **53**, 791–792 (1947).

The author proves by a category argument that if a formal power series with complex coefficients, $\sum a_{n_1\cdots n_k}z_1^{n_1}\cdots z_k^{n_k}$, is such that every parametrization $z_j=\sum_{n=1}^{\infty}c_{jn}t^n$ with complex c_{jn} leads to a power series with nonvanishing radius of convergence then the multiple series itself has a domain of absolute convergence. The problem is also raised of deciding the question for the case of real coefficients. *S. Bochner* (Princeton, N. J.).

Theory of Series

Mitrinovitch, Dragoslav S. Sur un procédé fournissant des solutions d'une équation aux différences finies rattachée à la théorie des coefficients de Stirling. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 244–247 (1947).

The Stirling numbers, defined by

$$(1)\qquad x(x-1)(x-2)\cdots(x-n+1)=\sum_{k=1}^{n}S_n^k x^k,$$

satisfy the recurrence relation

$$(2)\qquad S_{n+1}^m=S_n^{m-1}-nS_n^m.$$

The solution of (1) with $S_1^0=0$, $S_1^1=1$, is obtained with the aid of the numbers Φ_n^k which are defined by

$$(3)\qquad (x-1)(x-2)\cdots(x-n)=x^n+\sum_{k=1}^{n}(-1)^k\Phi_n^k x^{n-k},$$

and which satisfy the relation

$$(4)\qquad \Phi_n^k=n\Phi_{n-1}^{k-1}+(n-1)\Phi_{n-2}^{k-1}+\cdots+k\Phi_{k-1}^{k-1},\qquad k\leqq n.$$

From (3), $\Phi_n^1=\frac{1}{2}n(n+1)$; hence from (4) the numbers Φ_n^k of upper index k are determined from those of upper index $k-1$. Then S_n^{n-k}, $k=1, 2, \cdots, n-1$, are determined from the relation $\Phi_n^k=(-1)^k S_{n+1}^{n-k+1}$, which follows from the definitions (1) and (2). *D. Moskovits* (Pittsburgh, Pa.).

Hamming, R. W. Subseries of a monotone divergent series. Amer. Math. Monthly **54**, 462–463 (1947).

The author proves the following theorem. Let $I(n)$ be an increasing sequence of distinct positive integers, and let $\sum d_n$ be a monotone divergent series. A necessary and sufficient condition for the divergence of the subseries $\sum d_{I(n)}$ is that $I(n)/n$ is bounded. *T. Fort* (Athens, Ga.).

Dvoretzky, Aryeh, et Hanani (Chojnacki), Chaim. Sur les changements des signes des termes d'une série à termes complexes. C. R. Acad. Sci. Paris **225**, 516–518 (1947).

[In the original, the authors' names were misprinted as Arye Dvoretzky and Hanani Chojnacki.] The following

theorem, which is obvious when the a's are real and is more recondite in the general complex case, is proved. If $a_1+a_2+\cdots$ is a series of complex terms for which $\lim a_n=0$, then there is a sequence $\epsilon_1, \epsilon_2, \cdots$, of which each element is $+1$ or -1, such that the series $\epsilon_1 a_1+\epsilon_2 a_2+\cdots$ is convergent. The proof uses the Cauchy criterion for convergence and the following lemma. To each finite sequence $b_1, \cdots, b_N$ of complex numbers for which $|b_1| \leqq 1, \cdots, |b_N| \leqq 1$ corresponds a sequence $\epsilon_1, \cdots, \epsilon_N$, of which each element is $+1$ or -1, such that $|\epsilon_1 b_1+\cdots+\epsilon_n b_n|<3^{\frac{1}{2}}$, $n=1, \cdots, N$. The lemma is proved, and it is stated without proof that the constant $3^{\frac{1}{2}}$ cannot be replaced by a smaller constant. The theorem, the lemma (with $3^{\frac{1}{2}}$ replaced by larger constants for higher dimensions), and their proofs are easily extended to series of vectors in Euclidean n-space. *R. P. Agnew.*

Dvoretzky, Aryeh. Sur les suites d'exposants à densité supérieure finie. C. R. Acad. Sci. Paris **225**, 481–483 (1947).

Let $\{\lambda_n\}$ be a positive sequence increasing to infinity. Let us set $N(\lambda)=\sum_{\lambda_n<\lambda} 1$, $D(\lambda)=N(\lambda)/\lambda$, $\bar{D}(\lambda)=\lambda^{-1}\int_0^\lambda D(x)dx$, $D^*=\limsup D(\lambda)$ $(\lambda\to\infty)$, $\bar{D}^*=\limsup \bar{D}(\lambda)$. It is supposed that D^* is finite. Let us also set

$$\Lambda(z)=\prod(1+z^2/\lambda_n^2), \quad \Lambda^*=\limsup r^{-1}\log \Lambda(r).$$

The author proves that $D^*/e \leqq \bar{D}^* \leqq D^*$, $C_1 D^* \leqq \Lambda^* \leqq \pi D^*$ with $C_1=\max_{x>0} x^{-1}\log(1+x^2)=0.804\cdots$; $eC_1\bar{D}^*<\Lambda^*<\pi\bar{D}^*$. On setting $L(s)=\int_0^\infty e^{-sx}\Lambda(ix)dx$, the author proves that, for $0<D^*<\infty$, the points $s=\pm i\Lambda^*$ are the only singularities of $L(x)$ on $|s|=\Lambda^*$. If the λ_n are integers, we have also: $\varphi(D^*)D^* \leqq \bar{D}^* \leqq D^*$, where $\varphi(t)=(1-t)^{(1-t)/t}$.

S. Mandelbrojt (Paris).

Netanyahu, Elisha. Sur les moyennes de la densité d'une suite d'exposants. C. R. Acad. Sci. Paris **225**, 518–520 (1947).

Let $\{\lambda_n\}$ be a positive sequence increasing to infinity. Let us set $N(\lambda)=\sum_{\lambda_n<\lambda} 1$, $D_0(\lambda)=N(\lambda)/\lambda$, $D_n(\lambda)=\lambda^{-1}\int_0^\lambda D_{n-1}(x)dx$ $(n \geqq 1)$. The author proves the following theorem. If $D_m=\lim_{\lambda\to\infty} D_m(\lambda)$ exists for a certain m, then $\lim D_n(\lambda)$ exists for each n and this limit is equal to D_m.

S. Mandelbrojt (Paris).

Karamata, J. Sur la sommabilité de S. Bernstein et quelques procédés de sommation qui s'y rattachent. Rec. Math. [Mat. Sbornik] N.S. **21**(63), 13–24 (1947). (French. Russian summary)

The following methods of summability of a series $\sum u_\nu$, familiar in the theory of trigonometric series, are (mainly) discussed: the method

$$B_h: \sum_0^n \cos\left(\tfrac{1}{2}\pi\nu/(n+h)\right)u_\nu \to s, \quad n\to\infty,\ 0 \leqq h<1,$$

and its "continuous" analogue

$$\bar{B}: \sum_{\nu\leqq x} \cos\left(\tfrac{1}{2}\pi\nu/x\right)u_\nu \to s, \qquad x\to\infty.$$

It was proved by the reviewer that both $\bar{B}$ and B_0 are equivalent to the $(C, 1)$-method and that B_h is at least as effective as $(C, 1)$ [Math. Ann. **95**, 110–134 (1925); Math. Z. **25**, 132–149 (1926)]. It is also known that $B_{\frac{1}{2}}$ is actually more effective than $(C, 1)$ [S. Bernstein, C. R. Acad. Sci. Paris **191**, 976–979 (1930); P. Kharchiladzé, same Rec. **11** (**53**), 121–148 (1942), p. 145; these Rev. **7**, 59]. The main new result of the author is that B_h is equivalent to $(C, 1)$ whenever $|h-\frac{1}{2}|>c$, where

$$c=\min_{(\alpha)} 2\pi^{-1}\int_0^{\pi/2} |t\cos t-\alpha\sin t|\,t dt \quad (<.05);$$

and that $B_{\frac{1}{2}}$ is equivalent to $(C, 1)$ if, and only if, $u_\nu=o(\nu)$. The elementary methods of proof apply, with similar results, to more general methods of summation where the cosine is replaced by suitable other functions.

W. W. Rogosinski (Newcastle-upon-Tyne).

Delange, Hubert. Théorèmes taubériens généraux. II. C. R. Acad. Sci. Paris **225**, 483–485 (1947).

[For part I cf. the same vol., 28–31 (1947); these Rev. **9**, 28.] The author states a number of Tauberian theorems relating the asymptotic behaviour of $s(t)$ to that of $\int_0^\infty K(t/x)ds(t)$ and uses them to establish the following result. Let $f(z)$ be an integral function of order $p \geqq 1$ with only real negative zeros, and let $n(t)$ be the number of zeros with modulus not greater than t. Let $\rho(x)$ be a real function satisfying

$$p<\liminf_{x\to\infty}\rho(x) \leqq \limsup_{x\to\infty}\rho(x)<p+1$$

and $\lim_{x\to\infty}|\rho(kx)-\rho(x)|\log x=0$ uniformly for $\epsilon \leqq k \leqq 1/\epsilon$. Then if $|\theta|<\pi$ and

$$\log|f(re^{i\theta})| \sim Ar^{\rho(r)}\cos\theta\rho(r), \qquad r\to\infty,$$

and if there is no odd integer m satisfying

$$\tfrac{1}{2}m\pi/(p+1)<|\theta| \leqq \tfrac{1}{2}m\pi/\rho_1, \quad \rho_1=\liminf_{x\to\infty}\rho(x),$$

it follows that $n(t)\sim A\pi^{-1}t^{\rho(t)}\sin\pi\rho(t)$ as $t\to\infty$.

H. R. Pitt (Belfast).

Fourier Series and Generalizations, Integral Transforms

Anderson, Shirley K. Definite divergence of the conjugate Fourier series. Duke Math. J. **14**, 803–805 (1947).

If $f(x)$ is of period 2π and Lebesgue integrable, and $\psi(t)=f(x+t)-f(x-t)$, then, at a point x_0 where

$$\int_\epsilon^\delta \left|\frac{\psi(t)}{t}-\frac{\psi(t+2\epsilon)}{t+2\epsilon}\right|dt=O(1), \qquad \delta>0,$$

as $\epsilon\to 0$, the definite divergence to $+\infty$ $(-\infty)$ of the integral (conjugate function)

$$(2\pi)^{-1}\lim_{\epsilon\to 0}\int_\epsilon^\pi \psi(t)\cot\tfrac{1}{2}t dt$$

is necessary and sufficient for the definite divergence to the same value of the conjugate series associated with $f(x)$.

K. Chandrasekharan (Princeton, N. J.).

Wolf, František. Contributions to a theory of summability of trigonometric integrals. Univ. California Publ. Math. (N.S.) **1**, 159–227 (1947).

In this paper the author collects together many of the results contained in two papers [Proc. London Math. Soc. (2) **45**, 328–356 (1939); Acta Math. **74**, 65–100 (1941); these Rev. **1**, 225; **3**, 124]. The results of his paper in Acta Math. are used to get many theorems on trigonometric integrals corresponding to known theorems on trigonometric

series. The statements of all these theorems are very complicated and too long to be quoted here.

There are seven chapters. In chapter I he gives some preliminary results and in chapter II a theory of generalized integrals required in the sequel. In chapter III he gives two fundamental theorems. The first gives, under certain rather complex conditions, inversion formulae for trigonometric integrals and the second asserts that under the same conditions the difference between the given trigonometric integral and the trigonometric series of a certain function will be uniformly summable to zero throughout a given interval. Chapter IV deals with trigonometric integrals summable over sets of positive measure and extends Kuttner's theorem [J. London Math. Soc. 10, 131–135 (1935)] and some similar results of Marcinkiewicz and Zygmund. In chapter V he extends some theorems of Hille, Offord and Tamarkin [Bull. Amer. Math. Soc. 41, 427–436 (1935)]. In chapter VI he gives some results on the inversion of order of integration in a trigonometric integral equivalent to the integration of trigonometric series term by term. Finally in the last chapter there are some very general inversion theorems which the author asserts will yield every known inversion theorem for trigonometric integrals.

A. C. Offord (Newcastle-upon-Tyne).

Roettinger, Ida. A generalization of the finite Fourier transformation and applications. Quart. Appl. Math. 5, 298–319 (1947).

A formal extension of the Fourier sine- and cosine-series transformations and their inverses to more general orthonormal trigonometric-series transformations and their inverses is made. Transform relations for even order derivatives and for convolutions are stated. Methods for obtaining formal solutions of certain boundary-value problems in one and two independent variables are outlined, the differential equations treated being restricted to those free of odd-order derivatives with respect to the transformation variable. As an application, the method of solution of a problem in heat conduction is indicated. Short tables of series transforms and a nineteen item bibliography are included.

J. L. Barnes (Los Angeles, Calif.).

Minakshisundaram, S. Notes on Fourier expansions. II. J. London Math. Soc. 21 (1946), 264–267 (1947).

[For part I cf. the same J. 20, 148–153 (1945); these Rev. 8, 150.] Consider the eigenvalue problem

$$\partial^2\omega/\partial x_1^2+\cdots+\partial^2\omega/\partial x_k^2=-\mu\omega$$

in a bounded Euclidean domain D with a sufficiently smooth boundary B, and (what is significant) the boundary condition (*) $\omega(z)=\omega(x_1,\cdots,x_k)=0$ on B. Let $0<\mu_1\leqq\mu_2\leqq\cdots\to\infty$ be the eigenvalues and ω_n the eigenfunctions, and for a function $f(x)$ in L_2 consider the expansion $\sum a_n\omega_n$. The author's result is that the limit behavior of

$$\sum_{\mu_n\leqq R}(1-\mu_n/R)^\beta a_n\omega_n$$

as $R\to\infty$ is a purely local property, provided $\beta>\frac{1}{2}k$. The comparison with the corresponding result for multiple Fourier series on the entire torus $0\leqq x_j<2\pi$, $j=1,\cdots,k$, is as follows. On the torus, β can be lowered by $\frac{1}{2}$, that is, $\beta>\frac{1}{2}(k-1)$; $f(x)$ may belong to L_1 (instead of to L_2); and the boundary condition (*) does not really "include" the case of pure periodic functions.

S. Bochner.

Szász, Otto. On Möbius' inversion formula and closed sets of functions. Trans. Amer. Math. Soc. 62, 213–239 (1947).

Let $1=\lambda_1,\lambda_2,\cdots$ be a sequence such that the product of two λ's is again a λ. The author studies the transformation (1) $y(t)=Tx(t)=\sum a_n x(\lambda_n t)$. Conditions are given under which (1) has an inverse of the form (2) $x(t)=\sum b_n y(\lambda_n t)$. The simplest of these is: in the space of bounded functions (1) has a unique inverse (2) if $\sum_{n=2}^{\infty}|a_n|<|a_1|$. The work is based on the formal identity $\sum a_n\lambda_n^{-s}\sum b_m\lambda_m^{-s}=1$. A particular study is made of the case that $\{a_n\}$ and $\{\lambda_n\}$ are completely multiplicative, i.e., $a_m a_n=a_{mn}$, $\lambda_m\lambda_n=\lambda_{mn}$. In this case the unique formal inverse (2) is given by $b_n=\mu(n)a_n$, where $\mu(n)$ is the Möbius function.

If the series (1) and (2) are uniformly convergent, then the functions $\{x_n=x(\lambda_n t)\}$ and $\{y_n=y(\lambda_n t)\}$ determine the same closed linear manifold. Several applications of this remark are given, of which the following are typical. Every even continuous function vanishing at $t=0$ can be uniformly approximated by finite linear combinations of the functions $\varphi(nt)$, where $\varphi(t)$ is even and of period 2 and, in $0\leqq t\leqq 1$, $\varphi(t)=\sin\frac{1}{2}\pi t$ or $\varphi(t)=t$.

The completeness of $\{y(\lambda_n t)\}$ in L_r $(1\leqq r\leqq\infty)$ is discussed under various hypotheses on $x(t)$. In particular, $\{y(\lambda_n t)\}$ is complete in $L_r(a,b)$ if $\{x(\lambda_n t)\}$ is complete in $L_r(a,b)$ and if (i) $\sum_n a_n x(\lambda_m\lambda_n t)$ can be integrated term by term and (ii) $\sum_n a_n\xi(\lambda_m\lambda_n)=0$ for $m=1,2,\cdots$ implies $\xi(\lambda_m)=0$ $(m=1,2,\cdots)$. Many applications are given. Examples are: $\{nt-[nt]-\frac{1}{2}\}$ is complete in $L_r(0,\frac{1}{2})$ for $r>1$; $\{\operatorname{sgn}\sin n\pi t\}$ is complete in $L_r(0,1)$ for $r>1$. Every function continuous in $<0,1>$ can be uniformly approximated by linear combinations of $\{1,t,t^n(\log t^{-1})^\delta\}$ $(\delta>0)$. The same is true of $\{1,t,(\log 1/t)^{1+\beta}k_\beta(t)\}$ $(-1<\beta\leqq 2,\ k_\beta=\sum n^\beta t^n)$. The methods and results of this paper show many points of similarity with those of D. G. Bourgin [same Trans. 60, 478–518 (1946); these Rev. 8, 512].

W. H. J. Fuchs.

Ahiezer, N. I., and Babenko, K. I. On weighted polynomials of approximation to functions continuous on the whole real axis. Doklady Akad. Nauk SSSR (N.S.) 57, 315–318 (1947). (Russian)

The authors consider the space C_φ of functions $f(x)$, continuous on $(-\infty,\infty)$, with $\lim f(x)/\varphi(x)=0$, where $\varphi(x)$ is a continuous function such that $\varphi(x)\geqq 1$, $\lim x^n/\varphi(x)=0$ for $n=0,1,2,\cdots$. The norm in C_φ is $\sup|f(x)|/\varphi(x)$. They ask when the set $\{t^n\}_{n=0}^{\infty}$ spans C_φ (is "closed" in C_φ). They prove the following two theorems. (I) The set $\{t^n\}$ spans C_φ if there exists a sequence of polynomials $P_{n_k}(x)$ with $\varphi(x)\geqq P_{n_k}(x)>0$ and

$$(*)\qquad \lim\int_{-\infty}^{\infty}(1+x^2)^{-1}\log P_{n_k}(x)\,dx=\infty.$$

(II) The set $\{t^n\}$ does not span C_φ if

$$(**)\qquad \int_{-\infty}^{\infty}(1+x^2)^{-1}\log\varphi(x)\,dx<\infty.$$

By the usual consideration of linear functionals, (I) is reduced to showing that a moment problem of the form $\mu_n=\int_{-\infty}^{\infty}x^n\{\varphi(x)\}^{-1}d\sigma(x)$ is determined; this follows from (*) by means of a criterion of M. Riesz. The proof of (II) depends on a result of Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, pp. 16–18]: if the integral (**) converges, there is a function $F(x)$ vanishing for negative x and such that the absolute value of its Fourier

transform is $1/\varphi(x)$. Hence for some real $\psi(t)$

$$(\dagger) \qquad \int_{-\infty}^{\infty} \{\varphi(y)\}^{-1} e^{i\psi(y)} e^{ixy} dy = 0, \qquad x \leqq 0,$$

and because $\lim x^n/\varphi(x) = 0$ for every positive integer n, the integral in (†) can be differentiated, so that

$$\int_{-\infty}^{\infty} \{\varphi(y)\}^{-1} e^{i\psi(y)} y^n dy = 0 \quad (n = 0, 1, 2, \cdots);$$

on the other hand, for some $a > 0$,

$$\int_{-\infty}^{\infty} \{\varphi(y)\}^{-1} e^{i\psi(y)} e^{iay} dy = 2\pi F(a);$$

thus there is a linear functional on C_φ, vanishing for all x^n but not vanishing identically. *R. P. Boas, Jr.*

Babenko, K. I. On bases in Hilbert space. Doklady Akad. Nauk SSSR (N.S.) **57**, 427–430 (1947). (Russian)

Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, p. 100] gave a theorem which in the language of abstract Hilbert space reads as follows. If $\{x_k\}$ is a base and $\{y_k\}$ has the property that, for every sequence of constants a_k,

$$(*) \qquad \left\| \sum_{k=1}^{N} a_k (x_k - y_k) \right\| \leqq \theta \left\| \sum_{k=1}^{n} a_k x_k \right\|, \qquad 0 < \theta < 1,$$

then $\{y_k\}$ is a base. The author gives a short proof. [For other results in this direction and a bibliography, cf. Pollard, Ann. of Math. (2) **45**, 738–739 (1944); these Rev. **6**, 127.]

If $\{x_k\}$ is orthonormal, $\{y_k\}$ has the property that

$$(**) \qquad (1-\theta)^2 \sum |a_k|^2 \leqq \left\| \sum a_k y_k \right\|^2 \leqq (1+\theta)^2 \sum |a_k|^2.$$

The author shows that, if (**) is true, then $\{y_k\}$ can be connected by a finite chain of systems $\{x_k^{(1)}\}$ with an orthonormal set $\{x_k\}$ so that each of $\{y_k\}$ and $\{x_k^{(1)}\}$, $\{x_k^{(1)}\}$ and $\{x_k^{(2)}\}, \cdots, \{x_k^{(m)}\}$ and $\{x_k\}$ are connected by a relation of the form (*). He was evidently not aware that Duffin and Eachus [Bull. Amer. Math. Soc. **48**, 850–855 (1942); these Rev. **4**, 97] had proved that (**) implies the existence of an orthonormal set $\{x_k\}$ satisfying (*). *R. P. Boas, Jr.*

Harmonic Functions, Potential Theory

Nicolesco, Miron. Sur un critère d'harmonicité de Volterra et Vitali. C. R. Acad. Sci. Roum. **7**, 16–19 (1945).

The author proves the following analogue of a theorem due to Volterra and Vitali. Let D be a Dirichlet domain in the (x, y)-plane and let $u(x, y)$ be continuous in D. If for each $(x, y) \in D$ there exists a circle $C(x, y; \rho)$ in D with center (x, y) and radius $\rho = \rho(x, y)$ such that

$$u(x, y) = \pi^{-1}\rho^{-2} \int\int u(\xi, \eta) d\xi d\eta,$$

where the integral is extended over the interior of $C(x, y; \rho)$, then $u(x, y)$ is harmonic in D. In their theorem, Volterra and Vitali used a peripheral mean-value. *M. O. Reade.*

Hornich, H. Il primo problema al contorno per il piano a più tagli. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 63–67 (1947).

The author treats the Dirichlet problem for a plane slit region where the slits are finite in number and lie on a given line. Use is made of hyperelliptic integrals. *M. Heins.*

Ghizzetti, Aldo. Sopra un particolare problema misto di Dirichlet-Neumann per l'equazione di Laplace, trattato col metodo delle trasformate parziali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 131–168 (1946).

Let T be the plane region $0 < r < \infty$, $0 < \theta < \omega$, $\omega < 2\pi$. The author proves the existence of a harmonic function $u(r, \theta)$ in T satisfying the following conditions: (a) $u(r, 0) = f(r)$, $u_\theta(r, \omega) = g(r)$, $r > 0$, f and g given and continuous; (b) the integrals of $s^{\lambda-1}|f(s)|$ and of $s^{\lambda-1}|g(s)|$ are convergent at zero and the integrals of $s^{-(1/\lambda)-1}|f(s)|$ and of $s^{-(1/\lambda)-1}|g(s)|$ are convergent at infinity, where $\lambda = \pi/2\omega$;

$$(c) \qquad \lim_{r \to (\omega)0} \{\theta r^{(-)\lambda} u(r, \theta)/[\log 2\pi - \log(\omega - \theta)]\} = 0.$$

Uniqueness of the solution is proved under conditions (a) and (c). The proofs are made by expanding $u(r, \theta)$ in a Fourier series in $(0, \omega)$ and obtaining a Poisson integral-type formula for the solution of the problem.

J. W. Green (Los Angeles, Calif.).

Inoue, Masao. Sur les fonctions dont le logarithme est sousharmonique. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **3**, 1–13 (1943).

L'auteur reprend l'étude des fonctions $u(x, y) \geqq 0$ de classe PL, c'est-à-dire telles que $\log u$ soit sousharmonique (même au sens large comprenant la fonction $-\infty$). Il rappelle des critères de Montel, Radó, Beckenbach, comme celui que $ue^{\alpha x + \beta y}$ soit sousharmonique quels que soient α, β. Il reprend les démonstrations pour élargir les conditions suffisantes, d'une manière un peu compliquée mais qui permet, par exemple, dans l'énoncé rappelé, de remplacer $\alpha x + \beta y$ par $(x-\alpha)(y-\beta)$ ou $(x-\alpha)^2 - (y-\beta)^2$.

M. Brelot (Grenoble).

Inoue, Masao. Une étude sur les fonctions sousharmoniques et ses applications aux fonctions holomorphes. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **3**, 15–44 (1943).

L'auteur reprend d'abord pour s'en servir, après une petite extension, un énoncé de Brelot [Bull. Soc. Roy. Sci. Liége **8**, 385–391 (1939); ces Rev. **1**, 114] sur la majoration dans un cercle de centre O d'une fonction sousharmonique $u \leqq 0$ (majoration en M par la fonction de Wiener sur le cercle diminué du rayon opposé à OM, avec valeurs-frontière égales à 0 sur la circonférence, et ailleurs, à la distance r de O, à la borne inférieure de u sur la circonférence de rayon r). Il retrouve ou améliore des propositions de Beurling, en particulier une inégalité faisant intervenir la mesure logarithmique sur la droite [Beurling, thèse, Upsal, 1933]. Il en déduit une inégalité "fondamentale" qu'il utilise systématiquement pour donner sur les fonctions sousharmoniques des théorèmes du type Phragmén-Lindelöf. Il les applique aux fonctions holomorphes pour retrouver, étendre bien des théorèmes connus tel que celui de Wiman.

M. Brelot (Grenoble).

Mathéev, A. Sur une extension d'un théorème de W. Blaschke. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 201–212 (1946). (Bulgarian. French summary)

Après rappel de propriétés classiques sur les fonctions sousharmoniques et l'intégrale de Stieltjes, l'auteur donne un résultat qui revient aussitôt à ceci: si u est sousharmonique dans la sphère (cercle) unité de centre O, valant

$-\infty$ aux points de la suite P_k ($OP_k \neq 0$ et $\to 1$), harmonique hors des P_k, pour qu'il y ait une majorante harmonique, il faut et suffit que la série $\lambda_k G(O, P_k)$ converge, où λ_k est le coefficient du terme en $\log(1/P_kM)$ ou $1/P_kM^{n-2}$ dans le développement classique au voisinage de P_k. C'est là un résultat immédiat et qui paraît bien connu, dérivant du critère classique de majoration harmonique de u finie en O (critère contenu dans la répresentation de F. Riesz): convergence de $\int G(O, M)d\mu_M$ où μ est la distribution de masses correspondant à u. Mais l'auteur qui procède ainsi à la fin commence par utiliser le critère que les moyennes sur les circonférences (surfaces sphériques) concentriques sont bornées supérieurement. *M. Brelot* (Grenoble).

Sherman, D. I. Certain problems of the theory of potential. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 479–488 (1945). (Russian. English summary) [MF 15441]

The author shows how the potential function $u(x, y)$ which is regular in the finite simply-connected region S and which satisfies the boundary condition $a(s)u_x+b(s)u_y=f(s)$ on the continuously curved boundary L of S can be found as the solution of an equivalent Fredholm equation. Assuming $a^2(s)+b^2(s)=1$ all along L, two cases arise according to whether in the representation $a=\cos\omega(s)$, $b=\sin\omega(s)$ the function $\omega(s)$ increases by a nonpositive or by a positive multiple of 2π when the contour L is described once in the positive sense. In the first case the boundary-value problem always has a solution, in the second case there are non-trivial solutions of the homogeneous problem.

By the same method Poincaré's problem, where the boundary condition is $du/dn+p(s)du/ds+q(s)u=f(s)$, is reduced to a Fredholm equation, and so are certain singular integral equations. *M. Golomb* (Lafayette, Ind.).

Lagrange, René. Les familles de cônes de même sommet qui possèdent des harmoniques. Acta Math. **79**, 1–15 (1947).

The author starts from the equation of a family of cones of common apex and the corresponding equation of the orthogonal family of cones. Introducing polar coordinates corresponding to the common apex he obtains the Laplace partial differential equation in these coordinates. The conditions of orthogonality between the two families of cones lead to simple conditions for the two polar angles as dependent on two of the Cartesian coordinates. The author then derives the condition necessary if the solutions of the Laplace equation are to be separable in the three independent variables. This condition leads to two elliptic differential equations which are of the same form. The coefficients of this single differential equation depend on the family of cones under consideration and on the coordinate axes to which the family is referred. By further discussion of the equation considerable simplification is obtained, facilitating the discussion of its solutions. In this discussion several cases are considered leading to different classes of harmonics. All of these classes are well known by themselves and they include spherical harmonics, Lamé harmonics and their several special cases. *M. J. O. Strutt* (Eindhoven).

Somigliana, Carlo. Sviluppi in serie delle espressioni della gravità e determinazione gravimetrica delle costanti del geoide. Mem. Soc. Astr. Ital. (N.S.) **18**, 3–31 (1946).

Differential Equations

Smith-White, W. B. The elementary existence theorem for differential equations. J. Proc. Roy. Soc. New South Wales **80**, 203–207 (1947).

The author observes that the proofs in the texts "Theorie der Differentialgleichungen" by Bieberbach [1st ed., Berlin, 1923] and "Ordinary Differential Equations" by Ince [London, 1927] for the differentiability with respect to y_0 of a solution $y(x, y_0)$ of $y'=f(x, y)$, $y(x_0)=y_0$, where $\partial f/\partial y$ is continuous, are wrong. He then gives a proof which is similar to but somewhat less elegant than that in volume 2 of de la Vallée Poussin's "Cours d'Analyse" [Louvain, 1938; reprinted by Dover Publications, New York, 1946; these Rev. **8**, 15]. *N. Levinson* (Cambridge, Mass.).

Horvay, Gabriel. Rotor blade flapping motion. Quart. Appl. Math. **5**, 149–167 (1947).

The author starts by the derivation of the differential equation of the flapping motion of the rotor blades of a helicopter. This equation determines the flapping angle of the motion and has the form of a linear inhomogeneous differential equation of the second order, the coefficients of the first and zero order terms having periodic coefficients, as has the nonhomogeneous term. After reviewing briefly the known solutions of the corresponding homogeneous equation the author starts to develop his own solution of this equation. In this solution he first assumes the parameter occurring as a multiplier of the periodic terms to be zero. Thereupon an infinite trigonometric series solution is adopted in the case of a finite parameter, leading up to an infinite determinantal equation of familiar type. From this the characteristic exponents of the solutions may be determined numerically and this is shown by the author in an example. The stability of blade flapping motion is then discussed using "stability diagrams" of familiar type. The discussion is carried through up to numerical results. The author represents the functional relationship between the characteristic exponent and the parameters of the differential equation numerically in a diagram obtained by what is called the "twin-ripple method" of approximation. The stability and instability regions of the diagram are discussed extensively. Further diagrams are given showing instability troughs of the differential equation obtained by a "4-ripple approximation." Finally the author discusses the solution of the inhomogeneous differential equation and the problems of resonance involved. *M. J. O. Strutt* (Eindhoven).

Ikeda, Yosiro. Die Randwertaufgaben der linearen Integralgleichung. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **10**, 1–34 (1941).

Every boundary problem for the differential equation $\sum_{k=1}^{n} b_k v^{(n-k)}(x)=0$, $a \leqq x \leqq b$, can be reduced to the resolution of an integral equation

$$v(x)=U(x)+\int_a^b G(x, \xi)v(\xi)d\xi,$$

where $U(x)$ is the solution of a suitable boundary problem for the differential equation $\sum_{k=1}^{n} a_k u^{(n-k)}(x)=0$ and $G(x, \xi)$ is a Green's function whose expression is given by the author. *C. Miranda* (Naples).

Wintner, Aurel. On the normalization of characteristic differentials in continuous spectra. Physical Rev. (2) **72**, 516–517 (1947).

Referring to the equation (1) $\psi''+f^2(r)\psi=0$, in which $f(r)>0$, the note shows that if

$$\int^{\infty} |(f'/f^{\frac{1}{2}})'/f^{\frac{1}{2}}|dr < \infty,$$

the general solution of (1) is given by the real and imaginary parts of a solution which is asymptotically

$$c[f(r)]^{-\frac{1}{4}} \exp\left(i\int^{r} f(r)dr\right),$$

as $r \to \infty$. Thus all solutions are bounded as $r \to \infty$ if and only if $f(r)$ is bounded from zero, and the normalization $\int |\psi(r)|^2 dr = 1$ is possible if and only if $\int^{\infty} [f(r)]^{-1} dr < \infty$.

R. E. Langer (Madison, Wis.).

Bogdanov, Yu. S. On normal systems of Lyapunov. Doklady Akad. Nauk SSSR (N.S.) **57**, 215–217 (1947). (Russian)

Following Lyapunov [Liapounoff], the author considers normal systems of solutions of the linear differential equation $dX/dt = XP$, where the elements of P are continuous and bounded functions of t for $t_0 \leqq t < \infty$. A necessary and sufficient condition on C that $X_2 = CX_1$ be normal when X_1 is normal is given. A canonical form for $X(t_0)$ when X is normal is also given. *R. Bellman* (Princeton, N. J.).

Wasow, Wolfgang. On the asymptotic solution of the differential equation for small disturbances in a laminar flow. Proc. Nat. Acad. Sci. U. S. A. **33**, 232–234 (1947).

The note concerns the differential equation

$$y^{iv} + a_4 y''' + \sum_{k=0}^{2} [b_k \lambda^2 + a_k] y^{[k]} = 0 \tag{1}$$

in a domain (complex) in which the coefficients are analytic and b_2 has a zero of the first order. The complex parameter λ has constant argument. The domain $0 < |Q(x)| \leqq K$, with $Q(x) = \int [-b_2(x)]^{\frac{1}{2}} dx$, is divided into three subregions by the curves on which the real part of $\lambda Q(x)$ vanishes. The note states four theorems on asymptotic representations of solutions of (1) in these subregions. No proofs are given.

R. E. Langer (Madison, Wis.).

Manacorda, T. Sul comportamento asintotico degli integrali dell'equazione: $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ quando $\lim_{x \to +\infty} q(x) = +\infty$. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 537–541 (1947).

It is shown that the solutions of the equation of the title, in $x_0 \leqq x < \infty$, where $p(x)$ is a bounded continuous function of x, and $q(x)$ is a positive continuous function with $\lim q(x) = \infty$, possess an infinite set of real zeros in the interval (x_0, ∞) if $\exp(-\int_{x_0}^{x} p(t)dt) \leqq a < \infty$. The author then discusses the order of magnitude of the distance between consecutive zeros. *R. Bellman* (Princeton, N. J.).

Manacorda, T. Sul comportamento asintotico degli integrali dell'equazione: $y''(x) + p(x)y'(x) + q(x)y(x) = 0$ quando $\lim_{x \to +\infty} q(x) = +\infty$. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 752–757 (1947).

The author gives sufficient conditions that the maxima of y^2, where y satisfies the equation of the title, are decreasing and tend to zero as $x \to \infty$. These are that $0 \leqq p(x) \leqq c_1$, $q(x) > 0$, $\lim q(x) = \infty$, $q'(x) \geqq 0$ and $\log q(x)$ approaches ∞ "regularly," where the definition of "regular" approach is too complicated to give here. *R. Bellman.*

Teodorčik, K. The stability of harmonically autooscillating systems. Doklady Akad. Nauk SSSR (N.S.) **56**, 367–369 (1947). (Russian)

The author considers a system where the amplitude a, phase ϕ and frequency ω are connected by relations

$$da/dt = \omega a(\mu - 1), \quad d\phi/dt = \omega\psi,$$

where μ and ψ depend upon a, ω and a parameter ϵ. For a stationary solution we must have $\mu = 1$, $\psi = 0$. If ϵ is perturbed to $\epsilon + d\epsilon$, this yields the equation

$$\mu(a + da, \omega + d\omega, \epsilon + d\epsilon) = 1,$$

and correspondingly for ψ. Expanding in terms of the da, $d\omega$ and $d\epsilon$, and taking first order terms, conditions are derived which ensure the stability of the system with regard to changes in ϵ. *R. Bellman* (Princeton, N. J.).

Levinson, Norman. Perturbations of discontinuous solutions of nonlinear systems of differential equations. Proc. Nat. Acad. Sci. U. S. A. **33**, 214–218 (1947).

The system $\dot{x} = f \cdot \dot{u} + \phi$, $\epsilon\ddot{u} + g \cdot \dot{u} + h = 0$ is considered when the parameter ϵ tends to zero. The first equation can be a vector equation, but not the second one. (A generalization in this direction is briefly mentioned at the end of the paper.) The functions f, g, h and ϕ depend on the vector x, on u and on t and even possibly on the parameter ϵ provided that as ϵ tends to zero finite limiting values exist. The system considered includes van der Pol's equation (the usual parameter tends to infinity) provided the time scale is changed, and also equations of the Rayleigh type (after a preliminary differentiation).

For small ϵ the solutions of the system are compared with the solutions of the degenerate system obtained by letting $\epsilon = 0$. The solutions of the degenerate system are allowed to be discontinuous; they must satisfy, however, several conditions regulating their behavior at the discontinuities. The author states that if the degenerate system has a solution in $a \leqq t \leqq b$ and if the positive parameter ϵ is sufficiently small and the initial conditions of a solution of the given system are sufficiently close to those of the degenerate solution then the solution of the given system exists over the same range $a \leqq t \leqq b$ and tends to the degenerate solution as ϵ tends to zero. Furthermore the partial derivatives with respect to the initial values of x and u at $t = a$ and also $\dot{u}$ at $t = a$ tend to those of the degenerate solution. In particular, if the degenerate solution has a periodic solution whose Jacobian (with respect to the initial conditions) is distinct from zero, then the original system has a periodic solution for ϵ small enough. No proofs are given, but the rather lengthy description of what constitutes a solution of the degenerate system is given in full detail.

F. Bohnenblust (Pasadena, Calif.).

Dorodnicyn, A. A. Asymptotic solution of van der Pol's equation. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 313–328 (1947). (Russian. English summary)

The author considers the periodic solution to van der Pol's equation

$$p\,dp/dx - \nu(1 - x^2)p + x = 0, \qquad p = dx/dt, \tag{*}$$

for large values of the parameter ν. Four kinds of regions in the (x, p)-plane are defined in such a way that within each (*) may be solved using asymptotic series in powers of $1/\nu$. The regions overlap one another so that an arc of (*) may be pieced together around an entire half-cycle. Asymptotic

series are obtained for the period and amplitude of the periodic solution.

Using a similar procedure, but in the Liénard plane, J. Haag [Ann. Sci. École Norm. Sup. (3) **60**, 35–64, 65–111 (1943); **61**, 73–117 (1944); these Rev. **7**, 299] has already treated the equation $d^2x/dt^2+\nu f(x)dx/dt+x=0$. Dorodnicyn's regions I, II, III, IV correspond to neighborhoods of Haag's arcs of first kind, terminal points, arcs of second kind, and boundary points of first kind. The asymptotic series in I and IV are essentially those found by Haag in his corresponding regions. The present paper, although less general than Haag's, has the advantage of being much more concise. *E. Gilbert* (Cambridge, Mass.).

Turbovič, I. T. On a question concerning nonlinear systems with variable parameters. Doklady Akad. Nauk SSSR (N.S.) **57**, 351–352 (1947). (Russian)

The author considers the equation

$$\frac{d}{dt}\{\dot{q}|\dot{q}|^{a-1}f_1(t)\}+\dot{q}|\dot{q}|^{b-1}f_2(t)+q|q|^{c-1}f_3(t)=0,$$

where $\dot{q}=dq/dt$, and $df_n/dt \ll f_n(t)$, $n=1, 2, 3$. Using the above inequalities to make some approximations, the author determines the period and amplitude of the oscillation. The error involved in making these approximations is not determined. *R. Bellman* (Princeton, N. J.).

Wittich, Hans. Ganze transzendente Lösungen algebraischer Differentialgleichungen. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 71–73 (1946).

Let $w_i=d^iw/dz^i$ and for the monomial $w^{i_0}\cdots w_n^{i_n}$ define $i=i_0+\cdots+i_n$ and $g=0i_0+1i_1+\cdots+ni_n$ as its dimension and weight, respectively. For the nth order algebraic differential equation

$$(*) \qquad \sum a_{i_0\cdots i_n}(z)w^{i_0}\cdots w_n^{i_n}=0,$$

in which $a_{i_0\cdots i_n}(z)$ are polynomials in z, let $d=\max i$, where the i's are the dimensions of the monomial in (*). The author states the following results. If (*) contains only one term of maximum dimension d, then the differential equation has no integral transcendental (i.t.) solutions. This result still holds if the zero on the right side of (*) is replaced by an i.t. function of w. In case $n=1$, or (*) is of the form $a_0(z)w_n+\cdots+a_n(z)w+a_{n+1}(z)=0$, bounds for the order of growth of the i.t. solutions are given in terms of the weights as defined above and the degrees of the polynomials $a(z)$. The proofs are to appear later. The present paper proves the following theorem. In the differential equation $w_n+A_1w_{n-1}+A_2w=0$, let $A_1(z)$, $A_2(z)$ be integral functions. If this equation has a fundamental system $w_j(z)$ of i.t. solutions with finite orders of growths $\lambda_1, \cdots, \lambda_n$, then $A_1(z)$ and $A_2(z)$ must be polynomials. *F. G. Dressel.*

Bellman, Richard. The boundedness of solutions of infinite systems of linear differential equations. Duke Math. J. **14**, 695–706 (1947).

The paper deals with an infinite system of homogeneous linear differential equations with constant coefficients, $dx_i/dt=\sum_{k=1}^{\infty}a_{ik}x_k$, and solutions $x_i=x_i(t)$ satisfying $\sum_{i=1}^{\infty}|x_i(t)|^p<\infty$, where $p\geqq 1$ is given. In the case $p=2$, most of the results are known under more general conditions [cf. W. L. Hart, Amer. J. Math. **39**, 407–424 (1917) and for further references, A. Wintner, Amer. J. Math. **53**, 241–257 (1931)]. *P. Hartman* (Baltimore, Md.).

Žabotinskiĭ, M. E. On periodic solutions of nonlinear partial differential equations. Doklady Akad. Nauk SSSR (N.S.) **56**, 469–472 (1947). (Russian)

The author considers the nonlinear partial differential equation $L(y)-\rho y_{tt}=\mu f(y, x, \mu)$ with a boundary condition nonlinear in y, but involving the small parameter μ in the same way. By considering the solution of the linear equation $\mu=0$ and taking a solution of the above equation in the form of a power series in μ, the author obtains periodic solutions, using the classical method of Poincaré.

R. Bellman (Princeton, N. J.).

Picone, M. Sulla traduzione in equazione integrale lineare di prima specie dei problemi al contorno concernenti i sistemi di equazioni lineari a derivate parziali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 365–371 (1947).

Dans cette note et dans deux autres qui doivent suivre, l'auteur se propose d'étendre aux systèmes d'équations linéaires aux dérivées partielles une méthode de calcul numérique des solutions qu'il a déjà exposé plusieurs fois pour le cas d'une seule équation. On suppose ici un système de p équations avec p fonctions inconnues. Le premier membre de chaque équation, étant linéaire par rapport au groupe total des inconnues, est considéré comme la somme d'autant de polynômes différentiels linéaires dépendants chacun d'une seule inconnue. Appelant $E_{hk}(u_k)$ $(h, k=1, \cdots, p)$ ces polynômes partiels, le premier membre de la h-ème équation s'écrira donc $E_h(U)=\sum_k E_{hk}(u_k)$, en représentant par $U=(u_1, \cdots, u_p)$ l'ensemble des fonctions inconnues. L'auteur adopte une représentation vectorielle résumant le système donné sous la forme $E(U)=F$, où E est le complexe $E=(E_1, \cdots, E_p)$ ayant pour coordonnées les premiers membres des équations et $F=(F_1, \cdots, F_p)$ représente analoguement l'ensemble des deuxièmes membres. On indique alors par E_{hk}^* l'opérateur différentiel adjoint de E_{hk} et, indiquant par $V=(v_1, \cdots, v_p)$ un système de fonctions duquel on va disposer, on considère les expressions différentielles $E_k^*(V)=\sum_h E_{hk}^*(v_h)$ et le complexe $E^*(V)=(E_1^*, \cdots, E_p^*)$. Représentant par le signe $\times$ le produit scalaire, on forme ensuite l'expression $U\times E^*(V)-V\times E(U)$ dont l'intégrale, étendue sur un domaine donné D, par une transformation connue, se calcule par la connaissance des valeurs sur le contour des fonctions u_h et de certaines de leurs dérivées; étant d'autre côté $E(U)=F$, on déduit, comme d'ordinaire,

$$(1) \qquad \int_D U\times E^*(V)dT=\int_D F\times VdT+\Theta$$

où nous avons indiqué par Θ une intégrale sur le contour qu'on écrit facilement. En supposant de faire varier les fonctions v_h dans un système convenablement complet, l'auteur énonce en premier lieu un théorème d'unicité. On ne s'occupe pas de questions d'existence. En supposant que le système des vecteurs $\{E^*(V_\lambda)\}$ soit orthonormal, on déduit de (1) un développement de Fourier des fonctions u_h.

La partie du mémoire relative à ce dernier point est rédactée sous une forme complètement hypothétique et il semble à l'auteur de cette analyse qu'il ne manque pas quelque point obscur, comme la considération des fonctions $v_{\lambda l}$ complexes conjuguées des $w_{\lambda l}$, tandis qu'il n'y a pas de raison pour supposer, comme fait général, que celles-ci soient complexes. Aussi il semble que, pour que de l'égalité (1) on puisse isoler les différentes fonctions u_h, il sera nécessaire de donner à la propriété comprise sous le nom d'orthogonalité un sens tout particulier, qu'on peut d'autre côté préciser de différentes manières. *B. Levi* (Rosario).

Pleijel, Åke. Sur les opérateurs différentiels de type elliptique. Ark. Mat. Astr. Fys. **32A**, no. 14, 14 pp. (1946).

The paper deals with the linear partial differential equation

$$\sum_{i=1}^{N}\partial^2 u/\partial x_i^2 - q(x_1, \cdots, x_N)u + \lambda k(x_1, \cdots, x_N)u = 0$$

with any number N of independent variables. The matters considered in connection with it are those which were dealt with by the author in the case $N=2$ in an earlier paper [same Ark. **30A**, no. 21 (1944); these Rev. **6**, 228].

R. E. Langer (Madison, Wis.).

Trjitzinsky, W. J. Singular elliptic and hyperbolic partial differential equations. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 365–430 (1947). (English. Russian summary)

The author investigates the partial differential equation

$$(1) \qquad F(u) \equiv \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right) + C(x)u = f(x),$$

where x is the point $(x_1, \cdots, x_m)$ of the bounded domain D, embedded in the Euclidean m-dimensional space, and all the coefficients are assumed to be continuous in D and suitably differentiable; $A_{ij}=A_{ji}$. The technique developed is applied to the corresponding hyperbolic equation of normal type:

$$(2) \qquad F(u) = \rho(x)u_{tt} + \rho(x)f(x, t), \qquad \rho(x) > 0,$$

and $-\infty < t \leqq t_0$. The domain D can be highly irregular; the method is based on the use of approximating domains D_n of convenient regularity and such that $\lim D_n = D$. The coefficients are allowed to become unbounded in the neighborhood of the boundary $F(D)$ of D.

In a previous paper [Ann. of Math. (2) **43**, 1–55 (1942); these Rev. **3**, 244] the author has shown how to transform the differential equation considered into an integral equation of Fredholm type by making use of the adjoint operator to F. The major part of this paper is devoted to determining those conditions on the coefficients which will enable the differential equation to be presented as an integral equation and to enable spectral functions to be employed. The differential equations are assumed to be self-adjoint in general. For equations which are not self-adjoint a certain geodesic distance is used to advantage, depending on the quadratic form $\sum_{i,j=1}^{m} A_{ij}p_ip_j$. Explicit bounds for the coefficients are given in terms of this distance in the neighborhood of $F(D)$ in order that the differential equation considered can be transformed into an integral equation of Fredholm type. Solutions contained in L_2 are shown to exist and these solutions are given in terms of spectral functions when $f(x, t)$ is suitably restricted. The author indicates that the techniques employed were inspired by the work of Carleman [Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique, Uppsala, 1923; Ark. Mat. Astr. Fys. **24B**, no. 11 (1934)].

A. Gelbart (Princeton, N. J.).

Maĭnaradze, Leo. On the effective solution of the problem of Cauchy for certain linear partial differential equations of hyperbolic type. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoĭ SSR] **5**, 243–251 (1944). (Georgian and Russian) [MF 14602]

The solution of Cauchy's problem for the equation

$$u_{xx} + u_{yy} - u_{tt} + \lambda^2 u = 0,$$

λ = constant, with the conditions

$$u|_{t=0} = \varphi(x, y), \quad u_t|_{t=0} = \psi(x, y),$$

can be obtained as a certain integral transform of the solution of the same problem for the simpler equation

$$u_{xx} + u_{yy} - u_{tt} = 0.$$

Several such relationships are pointed out in this paper.

M. Golomb (Lafayette, Ind.).

Maĭnaradze, Leo. On a general representation of regular solutions of certain partial differential equations with imaginary characteristics. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoĭ SSR] **5**, 365–372 (1944). (Georgian and Russian) [MF 14604]

If in the equation of the elliptic type

$$(*) \qquad u_{xx} + u_{yy} + u_{zz} + \lambda^2 p(x, y, z)u = 0,$$

where λ is a constant and $p(x, y, z)$ is regular analytic in the neighborhood of some point (x_0, y_0, z_0), the substitution $t = iz$ is made, it becomes an equation of hyperbolic type with complex-valued coefficients. If in the usual way the Cauchy problem for the latter equation is reduced to an integral equation of Volterra's type, and if the above substitution is retraced, a Volterra equation in the complex z-plane is obtained whose solutions are the solutions of (*) which are regular in the neighborhood of the point (x_0, y_0, z_0). The author claims that this method can be successfully applied to differential equations of elliptic type of higher order and with any number of independent variables. No proofs nor details are offered.

M. Golomb (Lafayette, Ind.).

Maĭnaradze, Leo. On the asymptotic representation of solutions of certain linear partial differential equations of normal hyperbolic type for large values of the parameter. Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk Gruzinskoĭ SSR] **5**, 667–676 (1944). (Georgian and Russian) [MF 14610]

This paper deals with the asymptotic difference (for large values of λ) between the solution of Cauchy's problem for the equation

$$u_{xx} + u_{yy} - u_{tt} + \{p(x, y, t, \lambda) + \lambda^2\}u = 0$$

with the conditions $u|_{t=0} = \varphi(x, y, \lambda)$, $u_t|_{t=0} = \psi(x, y, \lambda)$, and the solution of the same problem for the equation

$$u_{xx} + u_{yy} - u_{tt} + \lambda^2 u = 0.$$

In estimating the difference use is made of a certain integral transform representation of the solution.

M. Golomb.

Dacev, Asen. On the cooling of bars composed of a finite number of homogeneous parts. Doklady Akad. Nauk SSSR (N.S.) **56**, 355–358 (1947). (Russian)

The mathematical problem reduces to the solution of the simultaneous partial differential equations

$$\partial^2 u_i/\partial x^2 = \partial u_i/\partial t,$$
$$i = 1, \cdots, n, \; x_{i-1} < x < x_i, \; 0 = x_1 < \cdots < x_n,$$

with the boundary conditions

(a) $u_i(x, 0) = \psi_i(x)$, $\quad x_{i-1} < x < x_i$;

(b) $u_1(x_0, t) = \phi_0(t)$, $\quad u_n(x_n, t) = \phi_n(t)$, $\quad t > 0$;

(c) $u_i = u_{i+1}$, $\quad x = x_i$, $t > 0$;

(d) $k_i \partial u_i/\partial x = k_{i+1}\partial u_{i+1}/\partial x$.

Denote $u_i(x_i, t)$ by $\phi_i(t)$. It is then possible by well-known methods to express the solution in the interval $x_i < x < x_{i+1}$, using boundary conditions (a) and (c), in terms of $\phi_i(t)$

and $\psi_i(x)$. Using (b), (d), a system of Volterra-type integral equations is obtained. These equations are similar to Abel's integral equation and may be solved similarly. The stationary case where $\phi_0(t)=c_0$, $\phi_n(t)=c_n$ is also treated as a special case. *R. Bellman* (Princeton, N. J.).

Dacev, Asen. On the cooling of a rod composed of two homogeneous rods of finite length. Doklady Akad. Nauk SSSR (N.S.) **56**, 255–258 (1947). (Russian)
The general case is reviewed above. *R. Bellman.*

Tranter, C. J. Heat flow in an infinite medium heated by a cylinder. Philos. Mag. (7) **38**, 131–134 (1947).
An infinite solid, initially at temperature zero, is heated by an infinitely long solid cylinder of another material. The cylinder is imbedded in the infinite solid and has a uniform positive initial temperature. A formula for the temperature distribution in the two media is determined by using the Laplace transformation. A simple approximate formula for the temperatures in the cylinder, when the thermal conductivity of the cylindrical medium is infinite, is determined. Numerical results are given and compared with those of S. Whitehead [Proc. Phys. Soc. **56**, 357–366 (1944); these Rev. **6**, 177]. *R. V. Churchill* (Ann Arbor, Mich.).

Walters, A. G. A problem on the conduction of heat. Philos. Mag. (7) **38**, 70–78 (1947).
An infinitely long hollow cylinder is surrounded by a thin coaxial tube with an air space between the two. The material in the two solids is not necessarily the same. The entire body is allowed to cool in air from an initially uniform temperature. One of the boundary conditions used by the author implies that the inner surface of the hollow cylinder is impervious to the transfer of heat. Radiation and convection at the other surfaces is allowed for by means of certain empirical laws. The author derives formulas for the temperatures $\delta(t)$ in the thin tube and $\theta(r,t)$ in the hollow cylinder, using series of Bessel functions.
R. V. Churchill (Ann Arbor, Mich.).

Kolesnikov, A. G. Calculation of the diurnal variation of temperature of the surface of the ocean. Doklady Akad. Nauk SSSR (N.S.) **57**, 149–152 (1947). (Russian)

Mendes, Marcel. Sur une équation aux dérivées partielles du troisième ordre. C. R. Acad. Sci. Paris **225**, 619–620 (1947).
The quantity $H=ad-g$ associated with the third order partial differential equation

$$(1)\qquad u_{xyz}+a(x)u_{yz}+d(y,z)u_x+g(x,y,z)u=0$$

remains invariant under certain types of transformations applied to (1). The author makes use of H to reduce (1) to systems of equations of order lower than three in case H satisfies such conditions as $H=0$, $H=f(x)$, etc. No proofs are given. *F. G. Dressel* (Durham, N. C.).

Integral Equations

Hellsten, Ulf. Determination of the denominator of Fredholm in some types of integral equations. Acta Math. **79**, 105–152 (1947).
Detailed proofs of results which were announced previously [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 118–122; these Rev. **8**, 518]. *M. Kac* (Ithaca, N. Y.).

Placzek, G., and Seidel, W. Milne's problem in transport theory. Physical Rev. (2) **72**, 550–555 (1947).
The authors modify the solution of the Milne integral equation given in Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, chap. 4] so that it will be suitable for applications to problems in the theory of neutron diffusion. *A. Heins* (Pittsburgh, Pa.).

Placzek, G. The angular distribution of neutrons emerging from a plane surface. Physical Rev. (2) **72**, 556–558 (1947).
The solution derived by Placzek and Seidel for the Milne equation [cf. the preceding review] is transformed into a form more suitable for numerical calculation. The numerical results found by the Mathematical Tables Project are given.
A. Heins (Pittsburgh, Pa.).

Mark, C. The neutron density near a plane surface. Physical Rev. (2) **72**, 558–564 (1947).
The solution of Milne's integral equation [cf. the second preceding review] is expressed as a real integral with a non-oscillating integral. The integrand contains the angular distribution of the neutrons emerging from the surface. The numerical work reported by Placzek [cf. the preceding review] is used in the numerical evaluation of this integral. The results are compared with some of the existing approximations. *A. Heins* (Pittsburgh, Pa.).

LeCaine, J. Application of a variational method to Milne's problem. Physical Rev. (2) **72**, 564–566 (1947).
An approximate solution of Milne's integral equation is obtained by a variational principle with good numerical accuracy and simple analytical form. *A. Heins.*

Mikhlin, S. G. Les théorèmes de Fredholm dans la théorie des équations intégrales singulières. C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 759–760 (1946).
In previous papers [Doklady Akad. Nauk SSSR (N.S.) **11** (1936 II), 3–6 (1936); same C. R. (N.S.) **19**, 353–355 (1938); Rec. Math. [Mat. Sbornik] N.S. **1(43)**, 535–552 (1936)] the author introduced the concept of the "symbol" of a singular integral equation. Subsequently he proved that a singular integral equation is equivalent to a Fredholm equation if and only if its symbol vanishes nowhere; operators having such a symbol are termed singular regular. It is proved that the three Fredholm theorems hold for singular regular equations; similar results hold for systems.
W. J. Trjitsinsky (Urbana, Ill.).

Gousseïnoff, A. Théorèmes d'existence et d'unicité pour les équations intégrales singulières non linéaires. Rec. Math. [Mat. Sbornik] N.S. **20(62)**, 293–310 (1947). (Russian. French summary)
The author studies the equation

$$(1)\qquad u(x)=\int_{-\pi}^{\pi}\varphi(x,s,u(s),\lambda)\cot\tfrac{1}{2}(s-x)ds$$

for small λ; φ is assumed periodic in x and s and $\partial\varphi/\partial u\equiv 0$ for $\lambda=0$;

$$(2)\qquad \begin{aligned}&|\varphi(x_2,s_2,u_2,\lambda)-\varphi(x_1,s_1,u_1,\lambda)|<k_1|x_2-x_1|^{h_1}\\ &\qquad+k_2|s_2-s_1|^{h_2}+k_3|u_2-u_1|=A(k_1,k_2,k_3);\\ &|\varphi_\lambda'(x_1,s_1,u_1,\tilde\lambda)-\varphi_\lambda'(x_1,s_1,u_1,\tilde\lambda)|<A(l_1,l_2,l_3)\end{aligned}$$

($\bar{\lambda}$ fixed). Using the method of fixed points [V. Niemytzki, same Rec. 41, 421–452 (1934), and Schauder, Studia Math. 2, 171–180 (1930)] existence of a solution of (1) is established. When, in addition, $|\varphi''_{\bar{\lambda}\lambda}|<M$ and $\varphi''_{\bar{\lambda}\lambda}$ satisfies a condition of form (2), a unique solution is obtained by successive approximations. These results are applied to conformal representation of the unit circle on a domain "near" the circle, the boundary of the domain being of the form $\rho=1+\lambda\psi(\theta)$ ($\psi(\theta)$ satisfying a Lipschitz condition).

W. J. Trjitzinsky (Urbana, Ill.).

Carafa, M. Risoluzione delle equazioni funzionali lineari nel campo analitico, mediante un numero finito di integrazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 521–527 (1947).

This is a summary of an investigation concerning the integral equation

$$\gamma(x,\lambda)-\frac{\lambda}{2\pi i}\oint_C \alpha^{-1}w(x,1/\alpha)\gamma(\alpha,\lambda)d\alpha=f(x),$$

where $f(x)$ and the kernel $w(x,\alpha)$ are analytic functions. [It is stated that the kernel must satisfy certain further conditions, but these are not given explicitly. It appears, however, that $w(x,\alpha)$ is assumed to be regular in the neighborhood of the point $x=0$, $\alpha=0$, and $f(x)$ to be regular in the neighborhood of the point $x=0$.] The principal result of the study is that the resolvent kernel $R(\lambda;x,\alpha)$, by means of which the solution can be written in the form

$$\gamma(x,\lambda)=f(x)+\frac{\lambda}{2\pi i}\oint_C \alpha^{-1}R(\lambda;x,1/\alpha)f(\alpha)d\alpha,$$

can be obtained by a process involving six definite integrations and one differentiation, starting with the given kernel w and two universal functions (functions which are independent of w and f, and which can be calculated once for all). The paper gives only a brief sketch of the complicated argument, which is based upon Fantappiè's theory of analytic functionals. A full exposition is to appear elsewhere.

L. A. MacColl (New York, N. Y.).

Parodi, Maurice. Solutions d'équations intégrales déduites de la solution d'équations fonctionnelles. C. R. Acad. Sci. Paris **222**, 633–635 (1946). [MF 16044]

By applying the Laplace-Carson transformation

$$\varphi(x)=x\int_0^\infty e^{-xt}f(t)dt$$

to functional equations whose solutions are known, integral equations with known solutions are obtained. For example, the functional equation $t^{-1}f(t^{-1})=f(t)$ has the solution $t^{-\frac{1}{2}}F(t^{\frac{1}{2}}+t^{-\frac{1}{2}})$, where $F(t)$ is an arbitrary function. Therefore, the integral equation

$$2x\int_0^\infty J_0(yx^{\frac{1}{2}})\varphi(\tfrac{1}{4}y^2)y^{-1}dy=\varphi(x)$$

has the solution

$$\varphi(x)=x\int_0^\infty e^{-xt}t^{-\frac{1}{2}}F(t^{\frac{1}{2}}+t^{-\frac{1}{2}})dt,$$

among which there is the particular solution $x^{\frac{1}{2}}$. Several such examples are listed in this paper, corresponding to various cases of the equation $t^{-\alpha}f(t^{-1})=f(t^\beta)$.

M. Golomb (Lafayette, Ind.).

Parodi, Maurice. Remarque sur certaines équations intégrales singulières résolubles par le calcul symbolique. C. R. Acad. Sci. Paris **225**, 661–662 (1947).

The author indicates a formal generalization of the convolution integral equation for the linear instrumentation problem and its known solution by the s-multiplied Laplace transformation. In the extension a more general type of kernel is used.

J. L. Barnes (Los Angeles, Calif.).

Parodi, Maurice. Sur un type d'équations intégro-différentielles singulières. Revue Sci. **85**, 233 (1947).

This note indicates the formal solution of the integro-differential equation obtained by taking the s-multiplied unilateral Laplace transform of the functional equation

$$f(z^2;m,n,\lambda)=2^\lambda(1+z^m)^n f(z;m,n,\lambda)$$

in which m, n, λ are positive integers. The method of solution is based on the transformation of a particular solution

$$f(z;m,n,\lambda)=(1-z^m)^n(\log z)^\lambda$$

of the functional equation for small values of λ, the use of a recursion relation in λ involving a product, and the use of the convolution theorem.

J. L. Barnes.

Calculus of Variations

Popoff, Kyrille. Sur une propriété des extrémales et le théorème de Jacobi. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **42**, 55–60 (1946). (Bulgarian. French summary)

Variational problems in the plane are considered. By holding the y-axis fixed and rotating the x-axis, all the extremals through a given point (taken to be (0, 0)) assume the same initial conditions; the Euler equations depend on $\tan(\alpha/2)$, where $\pi/2-\alpha$ is the angle between extremal and y-axis. The extremals, for analytic integrands, are expressible as power series in $\tan(\alpha/2)$, the coefficients being analytic functions $\eta_i(x)$. From knowledge of η_0 and η_1 the points conjugate to (0, 0) could be found. Problems in parametric form are similarly treated.

E. J. McShane.

Tonelli, L. Nuove ricerche su una speciale classe di problemi di calcolo delle variazioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1**, 247–250 (1946).

The author is interested in the existence of absolute minima of a functional

$$I(y(x))=\int_{x_0}^{x_1}\{f[y(x),y'(x)]/\varphi(y(x))\}dx$$

in two classes K and K_1 of admissible arcs. He assumes that φ has first and second derivatives in (Y_0,Y_1), has $\varphi(Y_1)=0$ and $\varphi(y)>0$ for $Y_0\leqq y<Y_1$; he further uses a continuous function ψ on (Y_0,Y_1) with $\psi(Y_1)<0$. The function f itself satisfies the two conditions: (1) $f_{y'y'}(y,y')>0$, and $f(y,y')-y'f_{y'}(y,y')\to 0$ as $y'\to+\infty$; (2) there are constants γ and $\bar{Y}$ such that $\gamma>0$, $Y_0\leqq\bar{Y}<Y_1$ and for which $f(y,y')\geqq 0$ if $y'\leqq 0$ and $f(y,y')\geqq\gamma y'$ for $y'>0$ provided $\bar{Y}\leqq y<Y_1$. He then defines K as the set of all absolutely continuous arcs passing through (x_0,Y_1) with $Y_0\leqq y(x)\leqq Y_1$ for x in (x_0,x_1), with $y'(x)\geqq\psi(y(x))$ almost everywhere in (x_0,x_1) and which make $f[y(x),y'(x)]/\varphi(y(x))$ integrable. In this class I has an absolute minimum and a minimizing arc has a continuous derivative on (x_0,x_1) such that $y'(x)\geqq\psi(y(x))$ everywhere in (x_0,x_1). He then alters his con-

dition (2) above on f, defines a new class of admissible arcs and again gets an existence theorem. The paper is merely an announcement of results and does not contain proofs.

H. H. Goldstine (Princeton, N. J.).

Sigalov, A. G. Sur les intégrales doubles du calcul des variations dans la forme paramétrique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 383–386 (1947).

The author discusses lower semi-continuity by considering the action of the integrand (which is defined as a convex, positively homogeneous, positive definite function on the bivectors) on polyhedral approximations. The result which he states generalizes a theorem of T. Radó [Trans. Amer. Math. Soc. **51**, 336–361 (1942); these Rev. **3**, 229].

H. Federer (Providence, R. I.).

Theory of Probability

Ottaviani, G. Su una equazione integrale della statistica matematica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 59–63 (1947).

Let X be a normally distributed random variable with mean zero and let Y depend on X so that the conditional distribution of Y for given X is normal with mean 0 and variance $f(X)$. The main purpose of the paper is said to be to draw attention to the importance of Cantelli's problem [Atti Accad. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **12**, 395–411 (1919)] to decide whether there exists a function $f(X)$ such that $X+Y$ is normally distributed. This leads to an integral equation for the moment generating function from which the author derives certain necessary properties of the hypothetical solution. *W. Feller.*

Kozulyaev, P. A. On a question of extrapolation of stationary random processes. Doklady Akad. Nauk SSSR (N.S.) **56**, 903–905 (1947). (Russian)

Let the random variables $\{x(t)\}$, $-\infty<t<\infty$, constitute a stationary stochastic process in the wide sense with $E\{x(t)\}=0$ and correlation function $R(t)$. The author considers the problem of minimizing $E\{[x(t+\lambda)-s(t)]^2\}$, where $s(t)=\sum_{k=1}^{n}a_k x(t-\eta_k)$, $0\leqq\eta_k\leqq T$, for given $\lambda>0$, $T\geqq 0$ and $n\geqq 1$. In the well-known case $R(t)=q^{|t|}$ $(0<q<1)$ the minimizing $s(t)$ is $q^{\lambda}x(t)$ which does not depend on n or T. The author's results are an extension of this case. Let $s^*(t)$ be the minimizing $s(t)$ if $a_2, \cdots, a_{n-1}$ are constrained to vanish, and if $\eta_1=0$, $\eta_n=T$. It is proved that if, for some values of a_1 and a_n, $a_1R(\eta)+a_nR(T-\eta)-R(\lambda+\eta)\equiv 0$ $(0<\eta<T)$ then $s(t)$ reduces to $s^*=a_1x(t)+a_nx(t-T)$; a_1 and a_n and the mean square error of the approximation are evaluated. The identity is satisfied if $R(t)$ is linear in the interval $[0, T+\lambda]$ and in this case the result reduces to one already discovered by the author [C. R. (Doklady) Akad. Sci. URSS (N.S.) **30**, 13–17 (1941); these Rev. **3**, 4]. *J. L. Doob.*

Kolmogorov, A. N., and Savost'yanov, B. A. The calculation of final probabilities for branching random processes. Doklady Akad. Nauk SSSR (N.S.) **56**, 783–786 (1947). (Russian)

Soit $P_k^{\alpha}(t)=P(T_k\to\alpha_1T_1+\cdots+\alpha_nT_n|t)$ la probabilité qu'une particule du type T_k donne, après K générations, α_1 particules du type $T_1, \cdots, \alpha_n$ particules du type T_n. La fonction génératrice $F_k(t;x)=\sum_{\alpha}P_k^{\alpha}(t)x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $F_k(1;x)=f_k(x)$, est donnée par

$$F_k(t+1;x)=f_k\{F_1(t;x), \cdots, F_n(t;x)\}.$$

On suppose, en introduisant s'il le faut un type fictif dont les particules demeurent invariables, que $f_k(0, \cdots, 0)=0$. Le groupe $T_{k_1}, \cdots, T_{k_m}$ est dit fermé si les particules qui lui appartiennent ne peuvent produire que des particules du même groupe; on suppose le système total indécomposable en deux groupes fermés. Un groupe est dit final si (a) il est fermé, (b) chacune de ses particules produit une particule exactement, (c) il ne contient aucun sous-groupe ayant les propriétés (a) et (b); à l'intérieur d'un groupe final les transformations constituent un cas particulièrement simple des chaînes de Markoff. Soit $\varphi_k(u_1, \cdots, u_s)=\sum q_k^{\beta}u_1^{\beta_1}\cdots u_s^{\beta_s}$ la fonction génératrice des $q_k^{\beta}=P(T_k\to\beta_1\Psi_1+\cdots+\beta_s\Psi_s|\infty)$ en décomposant le système total en groupes finals $\Psi_r=\{T_{r1}, \cdots, T_{rn_r}\}$, $r=1, \cdots, s$, et en types $T_{\theta_1}, \cdots, T_{\theta_n}$ n'appartenant pas à des groupes finals; si on écrit T_{rm} au lieu de T_k, on écrira φ_{rm} au lieu de φ_k et f_{rm} au lieu de f_k. Théorème. Les relations

$$\varphi_{\theta m}=f_{\theta m}(\varphi_1, \cdots, \varphi_n), \qquad m=1, \cdots, n_\theta;$$
$$\varphi_{rm}=u_r, \qquad 0\leqq u_r<1;\ m=1, \cdots, n_r;\ r=1, \cdots, s;$$
$$0\leqq\varphi_k<1, \qquad k=1, \cdots, n;$$

déterminent univoquement les valeurs des φ_k pour les u_r donnés. On montre sur un exemple, étudié en detail, comment le cas de t continu peut se ramener au cas discret.

M. Loève (New York, N. Y.).

Yaglom, A. M. Certain limit theorems of the theory of branching random processes. Doklady Akad. Nauk SSSR (N.S.) **56**, 795–798 (1947). (Russian)

On considère des objets de type T dont chacun se transforme, en une unité de temps, en m objets de même nature (descendants) avec la probabilité $p_m=P_m(1)$; on étudie le comportement asymptotique des probabilités $P_m(t)$ d'une telle transformation en t unités de temps, lorsque $t\to\infty$. On introduit la fonction génératrice $F(t,x)=\sum_{i=0}^{\infty}P_i(t)x^i$, $f(x)=F(1,x)$, et on utilise la relation fondamentale $F(t+1,x)=f\{F(t,x)\}$. Soient $a=f'(1)$, $b=f''(1)$, $c=f'''(1)$, $Q(t)=1-P_0(t)$; on exclut les cas triviaux correspondants à l'un des $p_k=1$. Théorème 1. Si $a<1$ et b est finie, la distribution $\{P_m(t)\}$ tend vers une distribution limite dont la fonction génératrice $F(x)$ satisfait à

$$F\{f(x)\}=aF(x)+(1-a), \quad F'(1)=1/K,$$
$$K=\lim_{t\to\infty}(1-f\{F(t,x)\})/a^t.$$

Soit $\eta_t=\mu_t/M^*(t)$, où μ_t est le nombre des descendants d'un objet et $M^*(t)=a^t/Q(t)$. Théorème 2. Si $a=1$, $b\neq 0$ et c finies, alors $P\{\eta_t<y|\eta_t>0\}$ tend vers 0 pour $y<0$ et $1-e^{-y}$ pour $y\geqq 0$. Théorème 3. Si $a>1$ et $b\neq a^2-a$ est finie, alors cette distribution conditionnelle tend vers une distribution limite dont la fonction caractéristique $\psi(\tau)$ satisfait à $f\{(1-\lambda)\psi(\tau)+\lambda\}=(1-\lambda)\psi(a\tau)+\lambda$, $\psi'(0)=i$.

M. Loève (New York, N. Y.).

Yaglom, A. M. The ergodic principle for Markov processes with stationary distributions. Doklady Akad. Nauk SSSR (N.S.) **56**, 347–349 (1947). (Russian)

The author considers Markov processes with transition probabilities $p(t,x,E)$ (the probability of a transition from x into the set E in time t). He supposes that there is a stationary absolute probability distribution, that is, a distribution $W(A)$ satisfying $W(E)=\int p(t,x,E)W(dA_x)$ for all E, and is interested in the asymptotic properties of $p(t,x,E)$ under these hypotheses. Previous authors have usually presupposed properties of $p(t,x,E)$ which could be used to prove the existence of stationary limiting distributions for

fixed x, $t\to\infty$. However, the reviewer [Trans. Amer. Math. Soc. 44, 87–150 (1938)], Yosida [Proc. Imp. Acad. Tokyo 16, 43–48 (1940); these Rev. 1, 343] and Kakutani [Proc. Imp. Acad. Tokyo 16, 49–54 (1940); these Rev. 1, 343] have also made the hypothesis of a stationary distribution, but without his further hypotheses have obtained weaker results. Under the further hypothesis that $p(t, x, E)$ is absolutely continuous with respect to $W(A)$, so that there is a density $p(t, x, y)$, and under the hypothesis that $p(t, x, y)$ is continuous and positive, it is stated that $\lim_{t\to\infty} p(t, x, E) = W(E)$ for all x, E. Proofs are omitted but the method is indicated; it goes back to an analysis of averaging used by Khintchine in statistical mechanics [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 7, 111–122 (1943); these Rev. 5, 167] which yields the following theorem. Let $W(E)$ be a stationary distribution for a Markov process, and suppose that there is a positive density of transition as above. Then if any initial distribution is assigned, there is a corresponding density of distribution $q(t, x)$ at time t, and if $\Phi(\xi)$ is any strictly convex function ($\xi \geqq 0$) the integral $\int \Phi[q(t, x)]W(dA_x)$ is monotone nonincreasing in t, and there is equality for two values t_0 and t_1 if and only if $q(t_1, x) \equiv q(t_0, x) \equiv 1$; the value of this integral has its absolute minimum for $q \equiv 1$. *J. L. Doob* (Urbana, Ill.).

Yaglom, A. M. On the statistical treatment of Brownian motion. Doklady Akad. Nauk SSSR (N.S.) 56, 691–694 (1947). (Russian)

Kolmogoroff [Math. Ann. 113, 766–772 (1936)] found conditions necessary and sufficient that a Markov process whose variables take on values in a Riemann manifold define a reversible process in the sense that, if $f(t, x, y)$ is the probability (density) of a transition from x to y in time $t>0$ and $g(t, y, x)$ the probability of a transition from y to x in time $-t$, then $f=g$. The author considers the corresponding problem in which the manifold of the variables represents the phase space of a physical system so that the forward and reverse transition probability densities are now $f(t, x, \dot{x}, y, \dot{y})$ and $g(t, y, \dot{y}, x, \dot{x})$. The appropriate definition of reversibility is now $f(t, x, \dot{x}, y, \dot{y}) = g(t, y, -\dot{y}, x, -\dot{x})$. He makes the usual Brownian motion hypothesis that in the Fokker-Planck equations the matrix of dispersion coefficients does not depend on the velocities and that the average motion coefficients depend linearly on the velocities. The Fokker-Planck equations are written in invariant form and the desired necessary and sufficient conditions become essentially that a certain symmetric tensor have a zero covariant derivative and that a certain vector be a gradient. The general form of the stationary limiting distribution ($t\to\infty$) is found, and turns out, if simple identifications are made, to be the canonical distribution of Gibbs.

J. L. Doob (Urbana, Ill.).

Kampé de Fériet, Joseph. Analyse harmonique des fonctions aléatoires strictement stationnaires. C. R. Acad. Sci. Paris 225, 623–624 (1947).

Let a probability measure be defined on an abstract ω space, and suppose that $T_t\omega$ is a group of measure preserving transformations of space into itself, with $T_{s+t} = T_sT_t$. Then, if $f(\omega)$ is a measurable function, $X(t) = f(T_t\omega)$ defines a stationary stochastic process (one parameter stationary family of random variables). The harmonic analysis of the process can be obtained in terms of the harmonic analysis of the unitary group of transformations in L_2 defined by $U_tg(\omega) = g(T_t\omega)$. This harmonic analysis of a stationary process was obtained by Cramér [Ark. Mat. Ast. Fys. 28B, no. 12 (1942); these Rev. 4, 13] without using the theory of groups of unitary transformations, but the relation between the two theories has also been noted by other authors: for example, see Kolmogoroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) 26, 6–9, 115–118 (1940); these Rev. 2, 220].

J. L. Doob (Urbana, Ill.).

Blanc-Lapierre, André, et Fortet, Robert. Les fonctions aléatoires stationnaires de plusieurs variables. Revue Sci. 85, 419–422 (1947).

A stationary (wide sense) stochastic process depending on two variables x, y is defined by a family of random variables $\{X(x, y)\}$ satisfying (we write M for the point (x, y)) the conditions (a) $E\{X(M)\}$ is independent of M, (b) $E\{X(M')\overline{X(M)}\}$ depends only on the vector $\overline{MM'}$. The harmonic analysis of the one-dimensional case is extended to this two-dimensional case. *J. L. Doob* (Urbana, Ill.).

Argañaraz, Carlos J. M. New methods in the calculation of a ballistic pattern. An. Soc. Ci. Argentina 143, 49–82 (1947). (Spanish)

This article records an address honoring and recounting work of C. Biggeri. The case of a discrete pattern of hits is related as usual to the idealized case of a continuous distribution following the Gaussian law, considering throughout dispersion along the line of fire only. The author discusses two standard methods for estimating the center of impact, that of the arithmetic mean and that of the mean deviation, where in the latter method an adjusted estimate of the mean is made after each round. The author concludes that, while in the former method the expected deviation of calculated from true mean is inversely proportional to the square root of the number m of rounds fired, in the latter it is inversely proportional to m itself. He states that he has prepared tables for groups of 5–10, 15–20 and 25 rounds, etc., indicating the proper weighting to be assigned to observations to achieve the desired approximation. Letting $\theta(t)$ designate the usual probability integral $(2/\pi)\int_0^t e^{-\tau^2}d\tau$, he is led to evaluate $\int_0^\infty \{e^{-3t^2}/[1-\theta^2(t)]\}dt$, the method of whose evaluation, using contour integrals, occupies most of the paper. The article concludes with several pages on general questions. *A. A. Bennett* (Providence, R. I.).

Mathematical Statistics

Pompilj, Giuseppe. Sulla regressione. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 5, 186–219 (1946).

A statistical variable in a space of k components $(x_1, \cdots, x_k)$ may be regarded as determining for a given $r<k$ a pair of complementary variables $(x_1, \cdots, x_r)$, $(x_{r+1}, \cdots, x_k)$ of which the first has on the second a regression manifold. Let $\phi(x_1, \cdots, x_k)$ be the continuous density function, and let $m_i(x_{r+1}, \cdots, x_k)$ be the first regression moment $(\int_{-\infty}^{\infty} x_i\phi dx_i)/(\int_{-\infty}^{\infty}\phi dx_i)$, $i=1, \cdots, r$. The author establishes the following two characteristic properties and develops some of their implications. First: the regression system $m_i(x_{r+1}, \cdots, x_k)$, $i=1, \cdots, r$, is the unique system of r continuous functions of the $k-r$ variables $x_{r+1}, \cdots, x_k$ which, whatever be the functions $g_i(x_{r+1}, \cdots, x_k)$, satisfy the relations $M([x_i - f_i(x_{r+1}, \cdots, x_k)]\cdot g_i(x_{r+1}, \cdots, x_k)) = 0$, $i=1, \cdots, r$, and this holds even if one requires the functions $g_i(x_{r+1}, \cdots, x_k)$ to be continuous with squares bounded

in the mean. Second: the regression system (given above) is the unique system of r continuous functions with squares bounded in the mean, of $k-r$ variables $x_{r+1}, \cdots, x_k$, which renders a minimum the index of correlation

$$\sigma^2_{f_1 \cdots f_r} = M([x_1 - f_1(x_{r+1}, \cdots, x_k)]^2) + \cdots + M([x_r - f_r(x_{r+1}, \cdots, x_k)]^2).$$

The author then defines independence, independence in mean and algebraic independence of given grade, and in terms of these introduces corresponding types of liberation, namely, processes by which a statistical variable y is found so that $y = f(x) + \bar{y}$ and $\bar{y}$ is appropriately independent of x. These notions are applied to partial correlation coefficients.

A. A. Bennett (Providence, R. I.).

Westenberg, J. Mathematics of pollen diagrams. I, II. Nederl. Akad. Wetensch., Proc. **50**, 509–520, 640–648 (1947).

Description of R. A. Fisher's test of significance for the difference between two Bernoulli variables with an application to pollen counts. *W. Feller* (Ithaca, N. Y.).

Kitagawa, Tosio, Huruya, Sigeru, and Yazima, Takesi. The probabilistic analysis of the time-series of rare events. I. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **2**, 151–204 (1942).

The authors analyze the times of earthquake shocks over a 30 year period to see if they can be considered as coming from a Poisson process, in particular whether the three successive 10 year means must be considered to be different. They find that the answer to the last question appears to be affirmative for most of the 20 cities considered. The test for difference of means is due to Pitman [Suppl. J. Roy. Statistical Soc. **4**, 119–130 (1937)]. *J. L. Doob.*

Chakrabarti, M. C. The moments and seminvariants of the mean square successive difference. Bull. Calcutta Math. Soc. **38**, 185–189 (1946).

The moments of the mean square successive difference in samples from normal were found by von Neumann, Kent, Bellinson and Hart by the direct method [Ann. Math. Statistics **12**, 153–162 (1941); these Rev. **3**, 7]. In the present paper, using material found in the papers of Williams [Ann. Math. Statistics **12**, 239–241 (1941); these Rev. **3**, 7] and von Neumann [Ann. Math. Statistics **12**, 367–395 (1941); these Rev. **4**, 21], the author finds the characteristic function for this statistic and, from it, expressions for the moments and seminvariants and recurrence relations among them. *C. C. Craig* (Ann Arbor, Mich.).

Hoel, Paul G. The efficiency of the mean moving range. Ann. Math. Statistics **17**, 475–482 (1946).

This paper is primarily concerned with the distribution of the mean moving range $w = \sum_{i=1}^{n-1} |x_{i+1} - x_i| / (n-1)$ when the likelihood function is given by

$$(2\pi\sigma^2)^{-n/2} \exp\left\{-\tfrac{1}{2}\sigma^{-2}\sum_{i=1}^{n}(x_i - m)^2\right\}.$$

The first two moments of w for a finite n are given and it is also shown that $n^{\frac{1}{2}}(\frac{1}{2}\pi^{\frac{1}{2}}w - \sigma)$ is asymptotically normally distributed with a zero mean; the (asymptotic) efficiency of $z = \frac{1}{2}\pi^{\frac{1}{2}}w$ as an estimate of σ is approximately .605, while that of $\frac{1}{2}\delta^2$ (where δ^2 is the mean square successive difference) as an estimate of σ^2 is $\frac{3}{4}$. Asymptotic normality of more general mean moving ranges is also proved.

The reason given for using nonefficient statistics ($\delta^2/2$, z) to estimate σ^2 or σ is the desire to "eliminate the influence of trends," although the results are only shown to hold for the trendless case. It would seem that, in general, $\delta^2/2$ and z would be biased in the presence of trend. Further discussion would have to be based on a more rigorous statement of what type of trend is postulated. [In the formula for μ'' (line 4, p. 479), σ^2 has been omitted. In the summary preceding the paper the letter w is used to denote z.]

L. Hurwicz (Ames, Iowa).

Quensel, Carl-Erik. The validity of the z-criterion when the variates are taken from different normal populations. Skand. Aktuarietidskr. **30**, 44–55 (1947).

The problem considered is the distribution of the variance ratio in a two-component analysis of variance in which it is not assumed that the normally distributed variables have a common variance. The method used is to study the characteristic function of the joint distribution of the three sums of squares into which the sum of squares of the recorded values is subdivided. This is obtained in its general form and then in it each variance is replaced by a common value plus a deviation. Assuming these deviations to be small the characteristic function is then formally expanded in a power series which is recognized as the characteristic function of a Romanowsky type III series extended to three variables. Two special cases are studied first: (a) all variances within the same class (column in the usual arrangement) are equal and (b) the classes all contain the same number of observations and all the variances in the same row are equal. In both cases an approximation to the variance of the variance ratio is obtained. In the general case also a similar approximation is found. Subject to certain assumptions, in case (a) the author concludes that the variance found will be greater than if all population variances are equal; in case (b) it will be less, and in the general case it will be about the same or even less. *C. C. Craig* (Ann Arbor, Mich.).

Lehmann, E. L. On families of admissible tests. Ann. Math. Statistics **18**, 97–104 (1947).

Let W be the set of all tests of a hypothesis H at the significance level ϵ, and, given two elements w' and w'' of W, write $w'Rw''$ if the power of w' is uniformly greater than or equal to that of w''. Then the family F is a subset of W such that: (a) for any w in W there exists a w' in F such that $w'Rw$, and (b) if w' and w'' are both in F neither $w'Rw''$ nor $w''Rw'$ can hold. A family F is constructed explicitly for a case where, as shown by Neyman [Conférences internationales des sciences mathématiques, Genève, Colloque consacré à la théorie des probabilités, VI, Actual. Sci. Indus., no. 739, Hermann, Paris, 1938, pp. 27–57; J. Neyman and E. S. Pearson, Statist. Res. Mem. London **2**, 25–57 (1938)], a type A_1 test of a simple hypothesis exists. Certain desirable properties of the type A_1 tests are demonstrated; in a more special case, with an a priori distribution of the unknown parameter assumed, the type A_1 test is shown to minimize the average (over the parameter space) type II error. *L. Hurwicz* (Ames, Iowa).

Mann, H. B., and Whitney, D. R. On a test of whether one of two random variables is stochastically larger than the other. Ann. Math. Statistics **18**, 50–60 (1947).

This is a contribution in the field of nonparametric statistical inference. A test of the composite hypothesis H_0 (that $f(a) = g(a)$ for all a, where f and g are continuous cumulative distribution functions) is shown to be consistent

with regard to the "one-sided" class $\mathfrak{A}$ of admissible alternatives (that $f(a)>g(a)$ for all a). The distribution (under H_0) of the (rank type) statistic U used in the test is given. The test appears to be more efficient with regard to $\mathfrak{A}$ than that proposed by Wald and Wolfowitz [same Ann. 11, 147–162 (1940); these Rev. 1, 348] which, however, is consistent with regard to a broader class of alternatives. Power functions of the two tests have not yet been obtained.

An equivalent test was proposed by Wilcoxon [Biometrics Bull. 1, 80–83 (1945)] who confined his treatment to the case of equal size samples and gave 3 points of the relevant distribution. The use of U was proposed and an (incorrect) distribution derived by Mathen [Sankhyā 7, 329 (1946)], but this work was not known to the authors until after completion of the present paper. [In equation (1), p. 51, M should be replaced by m.] *L. Hurwicz.*

Savage, L. J. A uniqueness theorem for unbiased sequential binomial estimation. Ann. Math. Statistics 18, 295–297 (1947).

In a sequential sampling procedure from a binomial population, a point (x, y) is said to be accessible if the procedure permits observing x nondefectives and y defectives, and if the procedure does not terminate at this stage. A procedure is said to be simple if for every n all points with integral coordinates on the line $x+y=n$ between any two accessible points are also accessible. Simplicity, previously shown to be necessary for the existence of a unique unbiased estimate for the parameter of the population [Girschick, Mosteller and Savage, same Ann. 17, 13–23 (1946); these Rev. 8, 477], is shown to be sufficient also. *D. Blackwell.*

Paulson, Edward. A note on the efficiency of the Wald sequential test. Ann. Math. Statistics 18, 447–450 (1947).

The expected number of observations for a Wald sequential test of the hypothesis $\theta_0=\theta_1$ against one-sided alternatives is shown rather generally to become independent of the form of the distribution function when $\theta_1-\theta_0$ approaches zero. Hence Wald's table [same Ann. 16, 117–186 (1945), p. 147; these Rev. 7, 131] of efficiencies for the normal distribution is more generally applicable when $\theta_1-\theta_0$ is small. *A. M. Mood* (Ames, Iowa).

Stein, Charles, and Wald, Abraham. Sequential confidence intervals for the mean of a normal distribution with known variance. Ann. Math. Statistics 18, 427–433 (1947).

This paper considers sequential procedures for obtaining confidence intervals of specified length for the mean of a normal distribution with known variance. It is shown that the usual nonsequential procedure is superior to any sequential procedure in the sense that the former minimizes the least upper bound of the expected number of observations. *A. M. Mood* (Ames, Iowa).

TOPOLOGY

Bernheim, Baruch. Partitions of convex polygons into pentagons. Riveon Lematematika 1, 95–98 (1947). (Hebrew)

A subdivision of a convex n-gon into k convex pentagons ($k>1$) is called regular if no vertex of one pentagon lies on an open edge of another. It is proved that the minimum value of k is $[n/3]$ for $n>5$, $12-n$ for $n=2, 3, 4$, and $11-n$ for $n=5, 0$. The same proposition was proved by P. Mahlo [Topologische Untersuchungen über Zerlegung in ebene und sphärische Polygone, Dissertation, Halle, 1908] without the regularity condition. *S. Eilenberg* (New York, N. Y.).

Efremovič, V. A. Regular polyhedra. Doklady Akad. Nauk SSSR (N.S.) 57, 223–226 (1947). (Russian)

A polyhedron (two dimensional) is called locally regular if the number of angles in each face and the number at each vertex are constant. These numbers, m and μ, respectively, determine a "local type." In a preliminary paragraph the author investigates the local types which are possible on a manifold of given Euler characteristic. The lemma is stated that two polyhedra of the same local type possess a common covering surface.

A polyhedron is called completely regular if it admits automorphisms carrying an arbitrary face into any other in each of $2m$ distinct ways. It seems very likely that the polyhedra must be thought of as manifolds endowed with a hyperbolic, Euclidean, or elliptic metric; this is nowhere explicitly stated. It is shown that there are only a finite number of regular polyhedra lying on a given surface of nonvanishing characteristic; the infinitely many possible types belonging to the torus and the Klein surface are completely described. The principal tool is the character of a factor group appearing in a natural way in the covering surface (elliptic, Euclidean, or hyperbolic) associated with the polyhedron. *L. Zippin* (Flushing, N. Y.).

Cvetkova, A. I. On a theorem of L. S. Pontryagin on the displacement of cycles. Doklady Akad. Nauk SSSR (N.S.) 57, 331–334 (1947). (Russian)

Let M^n denote an unbounded, oriented, n-dimensional h-manifold [reference to Seifert and Threlfall, "Topologie," Teubner, 1934, chap. X]; the number of simplexes is supposed infinite. Simplicial complexes and star complexes are, in general, infinite; coefficients are integers mod m. The intersection index for a simplicial complex and star complex is defined, in a natural way, when the corresponding geometrical intersection is finite. Let K denote a closed simplicial complex in M^n. The author proves the following theorems. (1) If C^{n-r} is a star cycle whose intersection with K is finite and if, moreover, the intersection-index of C^{n-r} with an arbitrary r-dimensional cycle of K is equal to zero, then there exists a cycle C_1^{n-r} lying in M^n-K such that $C_1^{n-r}-C^{n-r}$ bounds a finite chain. (2) If C^{n-r} is a star cycle whose intersection index with every finite r-dimensional cycle of K is equal to zero, then there exists a cycle C_1^{n-r} in M^n-K such that $C_1^{n-r}-C^{n-r}$ bounds an infinite chain.

The proofs make use of the theory of character groups applied to appropriate groups in K and in B^{n-r}, the star complex mod m made up of stars dual to simplexes of K. The groups relevant to the first theorem are L_∞, consisting of all r-dimensional simplicial subcomplexes of K, Z_∞^r comprising the cycles, H_∞^r generated by boundaries of infinite chains; with these are associated $\mathfrak{L}^{n-r}$ made up of the finite star subcomplexes of B^{n-r}, $\mathfrak{Z}^{n-r}$ the subset of relative cycles, and $\mathfrak{H}^{n-r}$ the further subset of relative boundaries. *L. Zippin* (Flushing, N. Y.).

Gul', I. M. Topological incidence formula. Doklady Akad. Nauk SSSR (N.S.) 56, 895–898 (1947). (Russian)

This note extends the Lefschetz fixed point relation [reference to S. Lefschetz, Trans. Amer. Math. Soc. 28,

1–49 (1926)] to a more general many-valued continuous map T of a closed orientable manifold M^n on itself, $m \in M^n \to T(m) \subset M^n$. Points m for which $m \subset T(m)$ are called the fixed points, or the incidence points. It is postulated of the transformation that the graph Γ of the mapping is the carrier of some cycle Γ^{n+p}. The graph Γ is a subset of $M^n \times M'^n$: in this topological product M'^n is a copy of M^n. To each cycle γ^p of M^n corresponds a cycle $T(\gamma^p)$ defined by means of the cycle $\Gamma^{n+p} = \Gamma^{n+p} \cdot [\gamma^p \times M^n]$ as the copy in M^n of the projection of Γ^{p+p} into M'^n. The important incidence cycle ξ^p is defined by means of the cycle $\Gamma^p = \Gamma^{n+p} \cdot \Gamma_0$, where Γ_0 is the cycle based on the graph of the identity map T_0 of M_n upon itself. Now let γ_λ^p be an independent homology basis for M^n, ${}^*\gamma^{i,n-p}$ the dual basis, and P_μ the μth Betti number. The author establishes the formula:

$$\xi^p \approx \sum_{\mu=0}^{n-p} (-1)^\mu \sum_{i=1}^{P_\mu} T(\gamma_i^\mu) \cdot {}^*\gamma^{i,n-\mu}.$$

This reduces to the Lefschetz formula when $p=0$.

L. Zippin (Flushing, N. Y.).

Každan, Ya. M. An example of an open mapping of a one-dimensional locally-connected continuum on a square. Doklady Akad. Nauk SSSR (N.S.) **56**, 339–342 (1947). (Russian)

The construction of the example indicated by the title of this paper begins with the plane "universal" continuous curve S_0 [reference to W. Sierpiński, C. R. Acad. Sci. Paris 162, 629–632 (1916)] and a continuous mapping of S_0 upon a square T_0. The essential property of the mapping appears to be no more than is achieved by the upper semi-continuous decomposition of S_0 in which the interior squares bounding complementary domains are regarded as "points," and designated the multiple points of the mapping $f(S_0) = T_0$.

The author forms the topological products $S = S_0 \times C$ and $T = T_0 \times C$, where C is the Cantor ternary set on the unit interval, and the natural mapping $F(S) = T$ derived from f. The mapping F is not open. By a carefully described topological mapping G of S onto itself, the author achieves a continuous mapping $PGF(S) = T_0$ which is open. Here P designates the projection of T onto T_0. The mapping G is defined as the limit of a sequence of topological transformations of T onto itself, in all of which the "slices" $T_0 \times z$, $z \in C$, are invariant. Each transformation of the sequence modifies the mapping F in the neighborhood of a "multiple" point; the nature of this mapping viewed locally at the multiple point is akin to a rotation which "damps out" quickly as one moves away from the point. The definition of G is the key construction in the paper. However, the set S is not locally connected, and the author indicates the remaining steps in the passage from the map $PGF(S) = T_0$ to a similar type of map defined on a locally connected set, not specified.

L. Zippin (Flushing, N. Y.).

Vaĭnšteĭn, I. A. On closed mappings of metric spaces. Doklady Akad. Nauk SSSR (N.S.) **57**, 319–321 (1947). (Russian)

The author states (in some cases, without proofs) a number of theorems concerning continuous mappings, of which the following are typical. (1) If f is a closed continuous map of a metric space X onto a metric space Y, then the inverse image $f^{-1}(y)$, for every $y \in Y$, has a compact boundary. (2) Let f be a compact closed continuous mapping (i.e., a closed continuous mapping such that the inverse image of every point is compact) of a metric space X onto a metric space Y. If Y is compact (locally compact), then X is compact (locally compact). (3) Let f be a closed continuous mapping of a metric space X onto a metric space Y. Then there exist absolute G_δ's M and N (i.e., metric spaces which are G_δ's in any metric space containing them homeomorphically) and a closed continuous mapping φ of M onto N, such that $X \subset M$, $Y \subset N$, and φ is an extension of f. (4) If X is a metric space which is an absolute G_δ and if Y is a metric space which is the image of X under a closed continuous mapping, then Y is an absolute G_δ. *E. Hewitt.*

Vaĭnšteĭn, I. A. On a problem of P. S. Aleksandrov. Doklady Akad. Nauk SSSR (N.S.) **57**, 431–434 (1947). (Russian)

The author considers continuous mappings of a topological space X onto a topological space Y, with a view to finding closed subspaces of X which map onto Y and on which the mapping has a simple form, as well as finding resolutions of the given mapping into products of simple mappings. He considers in detail the following problem of P. S. Aleksandrov: let f be a continuous mapping of the space X onto the space Y, where the dimension of Y is not lower than the dimension of X. Under what conditions does there exist a space Z, a continuous map φ of X onto Z, and a continuous map ψ of Z onto Y, such that φ does not raise dimension, $f = \psi\varphi$, and $\psi(y)$ is a finite set for every $y \in Y$? The author shows that in general this resolution is not possible. He proves the following theorem. Let f be a continuous mapping of the zero-dimensional space X onto the full n-dimensional space Y, such that for every closed subset F of X such that $f(F) = Y$, $F = X$. Let Y have the property that every closed nowhere dense subset of Y has dimension less than Y. Then there exists a space Z, a continuous mapping φ of X onto Z, and a continuous mapping ψ of Z onto Y, such that $f = \psi\varphi$, $\psi^{-1}(y)$ is a finite set for every $y \in Y$, and $\dim Z \leqq n-1$. *E. Hewitt* (Chicago, Ill.).

Katětov, Miroslav. On H-closed extensions of topological spaces. Časopis Pěst. Mat. Fys. **72**, 17–32 (1947). (English. Czech summary)

In an earlier paper [same Časopis **69**, 36–49 (1940); these Rev. **1**, 317] the author constructed, for any Hausdorff space P, an H-closed extension τP such that (i) P is dense in τP, (ii) a continuous mapping f of P into a Hausdorff space can always be extended to a mapping of part of τP onto $f(P)^-$. Here he proves that τP is characterized as follows: (i) τP is H-closed and P is an open set in τP, and $\tau P - P$ is discrete; (ii) if $F_1, \cdots, F_n$ are relatively closed in P and $F = F_1 \cap \cdots \cap F_n$ is nowhere dense in P then $F_1^- \cap \cdots \cap F_n^- = F$ in τP. Similar results about other types of extensions obtained by imposing conditions concerning "relative semiregularity" are proved. The relation of the present work to that of Stone, Čech, Wallman, Čech and Novak, and Fomin is clearly set forth. *R. Arens.*

White, Paul A. Regular transformations on generalized manifolds. Duke Math. J. **14**, 769–775 (1947).

The results of this paper center around the invariance of generalized n-manifolds [see E. G. Begle, Amer. J. Math. **64**, 553–574 (1941); these Rev. **4**, 87] under n-regular maps. The definitions and certain relevant propositions will also be found in the author's previous paper on regular transformations [Duke Math. J. **12**, 101–106 (1945); these Rev. **6**, 182]. A simple example [orthogonal projection of a square on an edge] shows that conditions other than regularity must be imposed in order to obtain invariance. These conditions take the form of restrictions on the inverse of a

point or the complement of such a set. The work is done using Vietoris homologies mod 2 so that the manifolds are required to be compacta. *A. D. Wallace.*

Steenrod, N. E. Products of cocycles and extensions of mappings. Ann. of Math. (2) **48**, 290–320 (1947).

Es sei K ein endliches m-dimensionales Polyeder, simplizial zerlegt. Die Aufgabe, alle Klassen homotoper Abbildungen von K in die n-Sphäre S^n aufzuzählen, d.h. auf algebraisch-topologische Eigenschaften von K zurückzuführen, wurde für $m=n$ von Hopf [Comment. Math. Helv. **5**, 39–54 (1933)], für $m=3$, $n=2$ von Pontrjagin [Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 331–363 (1941); diese Rev. **3**, 60] gelöst. In der vorliegenden Arbeit wird der Fall $m=n+1$ vollständig behandelt. Das Resultat beruht auf einer neuen kombinatorischen Operation in K: Für jede ganze Zahl $i\geqq 0$ wird je zwei Ketten u, v der Dimensionen p bzw. q als Produkt $u \cup_i v$ eine Kette der Dimension $p+q-i$ zugeordnet, wobei $u \cup_0 v$ das Alexandersche Produkt $u \cup v$ ist. Sind u und v Cozyklen, so braucht i.A. $u \cup_i v$ kein Cozyklus zu sein; hingegen ist für einen Cozyklus u das Quadrat $u \cup_i u$ ein Cozyklus (falls $p-i$ gerade ist, nur ein Cozyklus mod. 2). Bei geeigneten Koeffizienten bewirkt $u \cup_i u$ einen Homomorphismus Sq_i der Cohomologiegruppen $H^p(K)$ in $H^{2p-i}(K)$, welcher topologisch invariant ist.

Die Klassifikation der Abbildungen f von K in S^n ($m=n+1\geqq 3$) wird dann so beschrieben. Zunächst gehört zu f ein ganzzahliger Cozyklus $z^n(f)$, welcher angibt, mit welchem Grad jeder n-Zyklus von K in S^n abgebildet wird. Der Satz von Hopf [loc. cit.] besagt, dass 2 Abbildungen f und g dann und nur dann auf dem n-dimensionalen Gerüst K^n von K homotop sind, wenn $z^n(f)\sim z^n(g)$ ist. Ist dies der Fall, so kann man $f/K^n = g/K^n$ annehmen; für jedes $(n+1)$-Simplex s bestimmen f/s und g/s, als Abbildungen der obern bzw. untern Halbsphäre einer S^{n+1} aufgefasst, ein Element der Homotopiegruppe $\pi_{n+1}(S^n)$ (die für $n\geqq 3$ von der Ordnung 2 ist). Es sei $d^{n+1}(f, g)$ die so definierte Kette mit Koeffizienten aus $\pi_{n+1}(S^n)$. Satz: f und g sind dann und nur dann homotop, wenn es einen ganzzahligen Cozyklus e^{n-1} gibt, derart, dass $d^{n+1}(f, g)\sim e^{n-1} \cup_{n-3} e^{n-1}$ ist (mit naheliegender Multiplikationsvorschrift für die Koeffizienten). Da jeder n-Cozyklus als ein $z^n(f)$ und jeder $(n+1)$-Cozyklus als ein $d^{n+1}(f, g)$ auftritt, folgt: Zu jedem Paar bestehend aus einem Element von $H^n(K)$ und einem von $H^{n+1}(K)/Sq_{n-3}H^{n-1}(K)$ gehört genau eine Klasse von Abbildungen von K in S^n.

Dieser Homotopiesatz folgt aus einem Erweiterungssatz, dessen einfachster Fall so lautet. K sei $(n+2)$-dimensional. Zu einer Abbildung f von K^{n+1} in S^n gehört eine Kette $c^{n+2}(f)$ mit Koeffizienten aus $\pi_{n+1}(S^n)$ [vgl. Eilenberg, Ann. of Math. (2) **41**, 231–251 (1940); diese Rev. **1**, 222], das sog. "zweite Hindernis." Es wird gezeigt, dass $c^{n+2}(f)\sim z^n(f) \cup_{n-2} z^n(f)$ ist. Also lässt sich f/K^n dann und nur dann auf K erweitern, wenn $z^n(f) \cup_{n-2} z^n(f)\sim 0$ ist. *B. Eckmann* (Lausanne).

Hu, Ši Dzen. A theorem on the extension of homotopy. Doklady Akad. Nauk SSSR (N.S.) **57**, 231–234 (1947). (Russian)

Let X be an ANR (absolute neighborhood retract) and A a closed subset of X. The equivalence of the following four conditions is established. (1) For any topological space Y, any mapping $f: X\to Y$ and any homotopy $\Phi: A\times I\to Y$ of f on A, there is an extension $\Phi': X\times I\to Y$ (I = interval $0\leqq t\leqq 1$); (2) $X\times 0 \cup A\times I$ is a retract of $X\times I$; (3) $X\times 0 \cup A\times I$ is an ANR; (4) A is an ANR. *S. Eilenberg.*

GEOMETRY

✶**Merz, Karl. Vielflache aus Scheitelzellen und Hohlzellen mit Abbildungen und Netzen.** Beilage Jber. Naturforsch. Ges. Graubündens **76–77**, 164 pp. (1939).

A double-edge (Doppelstrecke) is defined as the intersection of two planes each of which is a face of a polyhedron on each side of the intersection. Vertical cells (Scheitelzellen) are two polyhedral cells which have a double-edge in common. A polyhedron may be developed into a plane net in many different ways. An inversion joint (Wendestrecke) of a one-sided polyhedron, associated with one of its developed plane nets, is an edge across which one passes from the upper face of the net to the lower face of the net. This paper is a descriptive account, illustrated with 300 figures, of polyhedra made of vertical cells and the various plane nets of which they may be constructed. Classifications are made in accordance with the number of double-edges and the number and position of the inversion joints. Among the vertical-cell polyhedra which are exhibited are those made by cutting a convex polyhedron by an arbitrary set of planes and omitting some of the regions so formed. In others, some of the cutting planes are prismatic bundles or radial sets of planes.

The following errors are noted: A triangle has been omitted from the net of the one-sided heptahedron, fig. 33; the net of fig. 46 shows an unnecessary division of one of the faces and, furthermore, it cannot be assembled in the arrangement shown. Later papers on the same subject by the author appeared in Comment. Math. Helv. **13**, 49–53 (1940); **14**, 134–137 (1942); Vierteljschr. Naturforsch. Ges. Zürich **85**, 51–57 (1940); **87**, 193–198 (1942); these Rev. **2**, 260; **3**, 299; **7**, 164. *M. Goldberg* (Washington, D. C.).

Gandini, Adriano. Contributo di alcune dimostrazioni di teoremi relativi alla geometria del triangolo. Period. Mat. (4) **25**, 44–57 (1947).

✶**Szökefalvi Nagy, Gyula. The theory of geometrical constructions.** Univ. Francisco-Josephina. Kolozsvár. Acta Sci. Math. Nat., no. **18**, viii+87 pp. (1943). (Hungarian)

A very concise and clearly arranged treatment of the main results of the theory of constructions with compasses and ruler (Gauss' theory of the division of the circle, the constructions of Mohr-Mascheroni and Steiner-Poncelet, etc.) and with other tools (gauge, ruler with parallel edges, etc.) as well as the proof and applications of Lindemann's general theorem. The book contains some simplifications compared with the usual treatments and some results which were not previously included in text-books. In particular, there is a very simple, elementary proof of the theorem of Kortum and Smith concerning the constructions with compasses and ruler with the aid of a given conic. *L. Fejes Tóth* (Budapest).

Greenwood, Thomas. Les caractères de la droite euclidienne. Rev. Trimest. Canad. **33**, 174–187 (1947).

Die historische Entwicklung der Nicht-Euclidischen Geometrie aus dem Axiomensystem von Euclid führte dazu,

den Begriff Gerade in der Euclidischen-, Riemannschen- und Lobatschefskyschen Geometrie für Gebilde mit verschiedenen Eigenschaften zu gebrauchen. Methodische Erwägungen lassen es dem Verfasser wünschenswert erscheinen, die Grundlegung der drei Geometrien so abzuändern, dass derselbe Begriff in allen Geometrien im selben Sinne gebraucht wird. Dies gelingt z.B. auf grund metrischer Begriffe, wenn man die Gerade als Linie kürzester Entfernung definiert und von da aus durch zusätzliche Forderungen die Aufspaltung in die Geraden der Euclidischen- und der Nichteuclidischen Geometrien vornimmt. Die Geradenaxiome Euclids, nämlich die eindeutige Bestimmtheit durch zwei Punkte und das Parallelenaxiom, erscheinen dann als Folge eines sog. "Postulates der verschwindenden Krümmung." Diese Systematik, die hier nur angedeutet wird, ist anderswo durchgeführt [Rev. Trimest. Canad. 29, 295–316 (1943); diese Rev. 5, 86].

R. Moufang (Frankfurt am Main).

*Stiefel, Eduard. **Lehrbuch der darstellenden Geometrie.** Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, 11. Mathematische Reihe, Band VI. Verlag Birkhäuser, Basel, 1947. 173 pp. 24.50 Swiss francs; bound, 28.50 Swiss francs.

This book is based on a course which has been given at the Eidgenössische Technische Hochschule in Zürich for ten years. It is divided into four parts. The three chapters of part I fill approximately half of the book and cover the elements of descriptive geometry. The first chapter discusses orthographic projection and its use in solving graphically problems of solid geometry. In the second chapter pictorial representation of spatial objects is accomplished by means of orthogonal axonometric projection. The third chapter deals with various curves and surfaces in space which occur frequently in technical applications. A few pages on map projection conclude part I.

The second half of the book, organized into parts II, III and IV, introduces the reader to the idea of geometrical mappings. The elementary projective theory of conics and, to some extent, of surfaces of second order is discussed in part II. In part III the author investigates the theory of correspondences which map straight lines on straight lines. A unified treatment is offered for central projection, general (oblique) axonometry and perspective.

A feature of the book is its presentation of perspective which is therefore reviewed in the following in greater detail. To construct a "perspective" any three concurrent lines are chosen as images of the three coordinate axes. Two sets of three points are then selected arbitrarily on these axes, one to represent the unit points, the other to represent the points at infinity. The invariance of the cross ratio then permits one to transfer the Cartesian coordinates of the object to the image. The author proves what he calls the fundamental theorem of descriptive geometry. This is a generalization of Pohlke's theorem. It asserts that every perspective can be obtained as a projective transformation of a central projection. The author employs the term perspective in a somewhat unusual meaning. Customarily the term perspective is used as synonymous with the term central projection, whereas in this book it stands for "linear mapping." The author's "fundamental theorem of descriptive geometry" has been proved by E. Kruppa; a detailed discussion may be found in the book of Müller and Kruppa [Die linearen Abbildungen, Deuticke, Wien, 1923, part II, chap. 3, pp. 165–185]. Part IV gives a brief outline of stereographic projection and presents the elements of crystallographic drawing and Mercator's projection.

The book is well written. The author is successful in offering a concise and lucid presentation of the subject. To accomplish conciseness he has omitted problems for practice and made free use of the methods of analytic geometry. For the same reason he does not attempt to give a well balanced introduction to projective geometry. Since the book is likely to stimulate its readers, it would have been desirable to add a detailed bibliography for their guidance. Printing and figures are both excellent; there are remarkably few misprints. *E. Lukacs* (Cincinnati, Ohio).

Convex Domains

Lázár, D. Sur l'approximation des courbes convexes par des polygones. Acta Univ. Szeged. Sect. Sci. Math. 11, 129–132 (1947).

The author proves the following theorem. To every closed convex curve there exists an inscribed n-sided polygon of area t_n and a circumscribed n-sided polygon of area T_n so that $(T_n-t_n)/T_n \leqq \sin^2(\pi/n)$. *P. Erdös.*

Rényi, Alfréd. Integral formulae in the theory of convex curves. Acta Univ. Szeged. Sect. Sci. Math. 11, 158–166 (1947).

Given a convex plane closed curve C, suppose its supporting lines shifted inwards by the same distance μ to positions l_μ. The common part of the negative half-planes of the lines l_μ is termed the internal parallel curve $C(\mu)$ at distance μ [see B. v. Sz. Nagy, same Acta 9, 253–257 (1939); these Rev. 1, 264]. The curve $C(\mu)$ shrinks to a point, or to a segment, when $\mu=\rho$, where ρ is the radius of the largest inscribed circle of C. Write $A(\mu)$ for the area enclosed by $C(\mu)$, $P(\mu)$ for its length, $\kappa(\mu)$ for its "characteristic." In the case in which C is a polygon, $C(\mu)$ is a polygon also and $\kappa(\mu)$ is defined as $2\sum \tan \frac{1}{2}\beta_k$, where the β_k denote the external angles of $C(\mu)$. As the author observes, $\kappa(\mu)$ is then an increasing function of μ, and $\kappa(\mu)\geqq 2\pi$, $\kappa(\mu)=-P'(\mu)$, $P(\mu)=-A'(\mu)$, where P', A' are derivatives of P, A almost everywhere. From these facts, which he extends from convex polygons to convex curves in general by passage to the limit, the author derives integral representations of P and A, from which he obtains in a few lines inequalities connecting A, P, κ, ρ. Some of these are well known, for instance Bonnesen's form of isoperimetric inequality; others, such as (1) $A(0)\leqq \rho P(\frac{1}{2}\rho)$, are new. The author generalizes his method by defining the internal parallel curve $C_1(\mu)$ of a convex curve C_1 relative to a convex curve C_2 and the inequalities then concern mixed areas and characteristics.

L. C. Young (Princeton, N. J.).

Gál, I. S. A theorem on convex curves. Acta Univ. Szeged. Sect. Sci. Math. 11, 167–168 (1947).

An independent proof (for convex polygons and thence by passage to the limit for convex curves) of a result of A. Rényi [preceding review, inequality (1)].

L. C. Young (Princeton, N. J.).

*Knothe, Herbert. **Verallgemeinerungen des Hauptsatzes der Brunn-Minkowskischen Theorie.** Ber. Math.-Tagung Tübingen 1946, pp. 91–92 (1947).

Let $\rho_K(x)$ be a function defined on the interior points x of every convex body K in E^n with the following properties:

(1) $\rho_K(x)$ depends continuously on K and x; (2) ρ_K is homogeneous of degree m (m a positive integer) in x in the sense that $\rho_{\lambda K}(\lambda x)=\lambda^m\rho_K(x)$; (3) if $K=(1-\theta)K_1+\theta K_2$, $0\leqq\theta\leqq 1$, then

$$\log \rho_K\{(1-\theta)x^1+\theta x^2\}\geqq(1-\theta)\log\rho_{K_1}(x^1)+\theta\log\rho_{K_2}(x^2),$$

where x^i is an arbitrary interior point of K_i. Then $(\int_K\rho_K dx)^{(n+m)^{-1}}$ is a concave function of θ. For $\rho\equiv 1$ this becomes the Brunn-Minkowski theorem. No proofs are given. *H. Busemann* (Los Angeles, Calif.).

Algebraic Geometry

Berzolari, Luigi. Sulla curva piana razionale del quarto ordine. Ann. Mat. Pura Appl. (4) **24**, 13–37 (1945).

Étude des quartiques planes unicursales à partir de leur représentation rationnelle, au moyen des combinants généralisés des trois polynômes coordonnées. Ce travail prolonge les mémoires de Coble [Amer. J. Math. **31**, 183–212 (1909)], Thomsen [Amer. J. Math. **32**, 207–234 (1910)], Rowe [Trans. Amer. Math. Soc. **12**, 295–310 (1911); **13**, 387–404 (1912)] et Moore et Neelley [Amer. J. Math. **50**, 467–472 (1928)]. Outre l'expression géométrique des cinq combinants élémentaires, l'étude de l'involution fondamentale dont les groupes sont apolaires à toutes les sections rectilignes, l'étude des cubiques premières osculantes et des coniques secondes osculantes mixtes, ainsi que l'application des résultats aux lemniscates projectives, ce mémoire contient un exposé tout à fait suggestif des propriétés d'un faisceau ponctuel de coniques covariant à la quartique C. Voici quelques éléments remarquables de ce faisceau: (a) la conique I passant par les six inflexions de C, (b) la conique T passant par les 8 contacts des tangentes doubles, (c) la conique N enveloppe des droites portant les inflexions des premières osculantes, (d) la conique S enveloppe des droites qui coupent C suivant un quaterne équianharmonique; cette conique est tangente aux six tangentes d'inflexion de C et coupe C suivant les deux groupes équianharmoniques de l'involution fondamentale; (e) la conique K ainsi définie: si quatre points de C constituent un groupe de l'involution fondamentale, les premières osculantes en ces points ont deux à deux six tangentes d'inflexion qui sont les côtés d'un quadrangle complet inscrit dans K. *L. Gauthier* (Nancy).

Gaeta, F. Sulle curve sghembe di residuale uno. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 78–81 (1947).

An irreducible nonsingular curve C in ordinary space is said to be of "residual 0" if it constitutes the complete simple intersection of two surfaces, and is said to be of "residual ρ," where $\rho>0$, if there exist two surfaces whose complete intersection consists of C and an irreducible nonsingular curve C_1 of residual $\rho-1$, each component being simple. The author proves that a necessary and sufficient condition that C is of residual 1 is that the ideal of homogeneous polynomials vanishing simply on C has a base consisting of three terms. *J. A. Todd.*

Manara, C. F. Esistenza topologica di diramazioni negative per le curve doppie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 67–71 (1947).

In the study of the branch curves of multiple planes certain limiting curves may be studied with advantage [Zariski, Amer. J. Math. **51**, 305–328 (1929); Turpin, Amer. J. Math. **59**, 529–544 (1937)]. Difficulties arise, especially when the limiting curve contains a double component. These difficulties can be overcome, the author claims, by the introduction of the topological concept of negative branching. The following theorem is illustrated. At the points common to the double and simple components into which a variable curve degenerates, the double component has negative branching. *D. Pedoe* (London).

Jones, R. R. Note on a bisectable curve of trisecants. J. London Math. Soc. **21** (1946), 248–250 (1947).

Verfasser betrachtet die Gesamtheit der rationalen Normkurven C^4 des R_4, die durch 6 gegebene Punkte gehen und eine gegebene Gerade treffen, die Sehne einer dieser C^4 ist. Die so definierten Normkurven erzeugen eine rationale Fläche 9. Grades F^9, die auf die Ebene abgebildet werden kann durch die Gesamtheit der Kurven 5. Grades, die durch 15 Schnittpunkte der 6 Tangenten eines Kegelschnitts und noch einen weiteren Punkt gehen. Mit Hilfe dieser ebenen Abbildung wird dann gezeigt, dass durch jeden Punkt der Fläche ein quadratischer Kegel von drei Mal schneidenden Geraden geht, auf dem die drei Mal geschnittene Kurve in 2 rationale Kurven 5. Grades zerfällt. Ein ähnliches Verhalten zeigt die Fläche 3. Grades F^3 des R_3 mit 4 Doppelpunkten, das Duale zur Steinerfläche. Der scheinbare Umriss dieser Fläche von einem ihrer Punkte ist nämlich ein Paar von Kegelschnitten. Da diese F^3 in ähnlicher Weise wie die F^9 durch die Gesamtheit der Kurven 3. Grades durch die Schnittpunkte von 4 Geraden auf die Ebene abgebildet wird, gelingt es die F^9 und die F^3 durch Zwischenschalten weiterer Flächen noch mehr einander zu nähern. *W. Burau* (Hamburg).

Nagell, Trygve. Sur la classification des cubiques planes du premier genre par des transformations birationnelles dans un domaine de rationalité quelconque. Nova Acta Soc. Sci. Upsaliensis (4) **12**, no. 8, 34 pp. (1941).

The author considers plane cubic curves of genus one over an arbitrary coefficient field Ω. Two such cubics C and C_1 (in homogeneous coordinates) are called "collinearly equivalent" in Ω if one may be transformed into the other by a linear transformation with coefficients in Ω. Similarly, if there exists a birational transformation with coefficients in Ω which carries C into C_1, then C and C_1 are called "equivalent" in Ω. It is proved that, if C and C_1, with coefficients in Ω, are equivalent in Ω, then they are collinearly equivalent in the field $\Omega(\chi, \chi_1)$, where χ and χ_1 are the coordinates of any point of inflection of C and C_1, respectively. The reduction of these cubics to a Weierstrass normal form is studied; the form and the requisite birational transformation are determined. Given two cubics C and C_1 over Ω, the problem is considered to determine whether or not they are equivalent in Ω. This problem is solved in the case where the two given cubics both have the Weierstrass normal form. *H. N. Shapiro* (Princeton, N. J.).

Fano, Gino. Le trasformazioni di contatto birazionali del piano. Comment. Math. Helv. **20**, 181–215 (1947).

Dans ce mémoire l'auteur prolonge des recherches antérieures relatives aux transformations de contact birationnelles du plan. Il commence par rattacher la détermination de ces transformations à la considération des systèmes homaloïdaux de ∞^2 courbes du plan, et indique un procédé de représentation (algébrique et biunivoque) des éléments de contact d'un plan dans un espace ponctuel S_3, imaginé par S. Lie, au moyen duquel le problème de la recherche

des transformations birationnelles du plan est ramené à celui de la recherche des transformations crémoniennes de S_3 transformant en elle-même l'équation pfaffienne $dz+xdy-ydx=0$ d'un complexe linéaire général, dans lequel, aux systèmes homaloïdaux du plan, correspondent des congruences du 1^{er} ordre de courbes rationnelles. Une nouvelle représentation des éléments linéaires d'un plan π peut être obtenue au moyen des points d'une variété M_3^5 d'un S_7 envisagée par F. Severi. L'auteur met cette représentation à la base de l'étude des systèmes homaloïdaux, et cela lui permet de présenter, au sujet de la variété M_3^5, des développements d'un puissant intérêt géométrique.

La variété $M_3^6 \equiv (M)$ contient deux congruences rectilignes dont les droites représentent respectivement les éléments linéaires du plan π issus d'un point ou situés sur une droite quelconque, chaque congruence étant décomposable en ∞^2 cubiques réglées R^3, appartenant chacune à un S_4, et ayant pour directrices les droites de l'autre congruence. Par tout point M de (M) passent une droite de chaque congruence, et ces deux droites déterminent le plan principal en M; en adjoignant aux éléments précédents les lignes principales (tangentes en chaque point au plan principal correspondant, et images des éléments de contact distribués sur une courbe quelconque de π), on obtient les notions fondamentales du mémoire. Le problème de la détermination des transformations de contact dans π est ainsi ramené, dans (M), à celui de la détermination des transformations ponctuelles qui changent les éléments plans principaux et les courbes principales en éléments et courbes analogues, les transformations de contact birationnelles provenant de transformations particulières de (M) du type précédent, dites principales. Aux systèmes homaloïdaux de courbes d'ordre n et de classe ν de π correspondent, dans (M), des congruences Ω du 1^{er} ordre de lignes principales ω d'ordre $n+\nu$: les transformations principales de (M) sont celles qui changent les congruences Ω en congruences Ω.

Un réseau homaloïdal d'une Ω étant défini comme un système de ∞^2 surfaces de Ω formant un réseau homaloïdal lorsqu'on regard Ω comme une variété rationnelle ∞^2, l'auteur introduit les couples $[\Omega, \Omega']$ conjugués par la propriété que les courbes de chacune des Ω s'appuyant sur une même courbe de l'autre admettent pour lieu géométrique les surfaces d'un réseau homaloïdal, et déduit de là une nouvelle caractérisation des transformations principales de (M). La considération des directrices de deux Ω conjuguées non simultanément rectilignes conduit, en particulier, à la conclusion suivante: ou bien les courbes de l'une (seulement) des deux Ω déterminent sur les directrices (non communes aux deux) de l'autre des groupes d'une série linéaire ∞^2, ou bien les deux congruences sont contenues dans un système ∞^3 birationnellement équivalent à un complexe linéaire de S_3.

En ce qui concerne les congruences conjuguées, l'auteur, par l'emploi des transformations dites élémentaires (transformations principales transformant en elle-même l'une des deux congruences de droites de M) et de leurs produits, montre l'existence de suites limitées de congruences Ω, chacune étant conjuguée de la précédente et la dernière étant rectiligne. Toute Ω a une infinité de conjuguées dont l'ordre des courbes peut être aussi élevé que l'on veut. Celle ou celles de ces conjuguées dont l'ordre des courbes est minimum (congruences conjuguées minima) sont particulièrement intéressantes. L'auteur donne à leur sujet des aperçus susceptibles de provoquer de nouvelles recherches, et montre également l'intérêt qui s'attache aux propriétés des couples conjugués $[\Omega, \Omega']$ que l'on peut déduire des correspondances principales définies en rapportant Ω, Ω' respectivement aux congruences rectilignes de (M).

P. Vincensini (Besançon).

Differential Geometry

***Bödewadt, U. T. Vektoroperatoren in der Flächengeometrie.** Ber. Math.-Tagung Tübingen 1946, pp. 46–47 (1947).

Let $\mathfrak{n}(u, v)$ denote the unit vector in the direction of the normal of the surface $\mathfrak{x}(u, v)$ in E^3. The author observes that introduction of the operator

$$\mathfrak{c} = \mathfrak{n} \times \nabla = (EG-F^2)^{-\frac{1}{2}}(\mathfrak{x}_u D_u - \mathfrak{x}_v D_v)$$

allows one to simplify many proofs and formulas in classical differential geometry, for instance

$$K(EG-F^2)^{\frac{1}{2}} = |\mathfrak{n}, \mathfrak{c}\times\mathfrak{x}_u, \mathfrak{c}\times\mathfrak{x}_v|.$$

H. Busemann (Los Angeles, Calif.).

Hsiung, C. C. On triplets of plane curvilinear elements with a common singular point. Quart. J. Math., Oxford Ser. **18**, 129–132 (1947).

If two plane curves have a common point O, a common tangent at the point and singularities of the same kind at the point, so that the power series expansions for nonhomogeneous projective coordinates representing these two curves in the neighborhood of O are $y=ax^2+\cdots$ and $y=bx^2+\cdots$, $a\neq b$, the ratio a/b is Segre's projective invariant of the two curves [C. Segre, Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 6_2, 168–175 (1897)]. No such projective invariant exists for two curves at a common singular point if the singularities on the two curves are of different kinds. The author obtains a projective invariant for three plane curves having a common singular point O and a common tangent at the point, the singularities being of different kinds on the curves. He gives simple geometric characterizations of this invariant by means of cross-ratios for four cases of combined types of singularities. Three exponents λ, μ, ν determine the types of singularity on the three curves at O. The four cases are defined by the following conditions, respectively: (i) λ, μ, ν are integers; (ii) λ, μ, ν are rational fractions; (iii) two of λ, μ, ν are integers and the remaining one is a rational fraction; (iv) one of λ, μ, ν is an integer and the other two are rational fractions. *P. O. Bell.*

Su, Buchin. Alcuni invarianti di contatto di due varietà in uno spazio proiettivo. Boll. Un. Mat. Ital. (3) **2**, 9–12 (1947).

This is an extension to two m-dimensional varieties in contact at a point of a theorem of B. Segre [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **22**, 392–399 (1935)]. The author exhibits $k-1$ projective invariants of contact when the two m-varieties have in their common point the same r-osculating spaces for $r=1, 2, \cdots, k$, each of these osculating spaces having its maximal dimension. Their geometrical significance for $m>1$ is not determined.

J. L. Vanderslice (College Park, Md.).

Segre, Beniamino. Sul contatto di due varietà. Boll. Un. Mat. Ital. (3) **2**, 12–16 (1947).

Consider two analytic m-varieties in projective S_n tangent in a point O and admitting there common r- and $(r+1)$-osculating spaces S_a, S_b ($r\geqq 1$, $m\leqq a<b\leqq n$). When $b-a$ has its generic value the neighborhoods of order $r+1$ on the

two m-varieties determine a projectivity in S_b of center S_a together with a set of $b-a$ cones of vertex O and order $r+1$ in the tangent hyperplane at O. The projective invariants of projectivity and cones are projective invariants of contact of the two m-varieties at O. Their relation to the invariants of Su [see the preceding review] is noted and the latters' geometrical significance clarified. *J. L. Vanderslice.*

Doyle, Thomas C. Tensor theory of invariants for the projective differential geometry of a ruled surface. Duke Math. J. **14**, 381–399 (1947).

En vue d'une étude tensorielle des complexes de droites, l'auteur se propose d'appliquer d'abord les méthodes tensorielles à l'étude projective des surfaces réglées. Le point de départ est le système d'équations (1) $y_i''+p_i^r y_r'+q_i^r y_r=0$ $(i=1, 2)$ qu'il étudie par la méthode de Wilczynski par rapport aux transformations (2) $y_i=a_i^r y_r^*$, $t^*=\xi(t)$. Un vecteur relatif covariant de poids (k, w) se transforme d'après la loi $\lambda_i=a^k(\xi')^w a_i^r\lambda_r^*$, $a=|a_i^k|$, et analoguement les vecteurs contrevariants et les tenseurs. A l'aide du tenseur $U_j^k=u_j^k-\delta_j^k u_r^r/2$ $(u_i^k=p_i^{k'}/2+p_i^r p_r^k/4-q_i^k)$, l'auteur définit la différentielle covariante d'un vecteur

$$\lambda_{j.}=U\lambda_j'+Up_j^r\lambda_r/2+(1-2w+2k)U'\lambda_j/8+kUp_r^r\lambda_j/2, \qquad U=|U_j^k|.$$

C'est un vecteur de poids $(k, w+5)$. Il écrit ensuite la forme tensorielle du système (1) $y_{i..}=(U^2U_i^r+\rho\delta_i^r)y_r$, $\rho=UU''/8-(9/64)(U')^2+U^2u_r^r/2$, et ses formes canoniques d'après Wilczynski. Avec ces notations, les invariants et les covariants de Wilczynski et le système adjoint sont déterminés. Ensuite l'auteur prend un repère spécial associé à une génératrice et étudie des éléments de la surface définis au voisinage de cette génératrice: le plan tangent, l'hyperboloïde osculateur, les courbes flecnodales, les congruences flecnodales, le complexe linéaire osculateur, les surfaces flecnodales. Enfin, des conditions pour qu'une courbe flecnodale soit plane [Carpenter, Trans. Amer. Math. Soc. **16**, 509–532 (1915)] et des conditions nécessaires et suffisantes pour l'équivalence de deux surfaces [Cramlet, Trans. Amer. Math. Soc. **34**, 626–644 (1932)] sont déduites avec les nouvelles notations. *M. Haimovici* (Jassy).

Marcus, F. Sopra una classe di congruenze di rette proiettivamente applicabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 764–766 (1947).

A demonstration, based on an unpublished paper, is given of the existence of projectively applicable congruences as defined by Fubini and Čech [Geometria Proiettiva Differenziale, vol. 2, Zanichelli, Bologna, 1927, pp. 681–682]. *V. G. Grove* (Río Piedras, P. R.).

Bam-Zelikovič, G. M. On focal surfaces of stratified congruences. Doklady Akad. Nauk SSSR (N.S.) **56**, 671–674 (1947). (Russian)

G. Fubini [Ann. Mat. Pura Appl. (4) **1**, 241–257 (1924)] has studied pairs of congruences K, K' in one to one correspondence such that there exists a system of ∞^1 surfaces S which at their points of intersection with a ray r of K have tangent planes which all pass through the corresponding ray r' of K', and vice versa, with ∞^1 surfaces S' intersecting r' in points where the tangent planes pass through r. Such a pair of congruences are called stratified. In this paper it is shown that it is possible to connect with an arbitrary surface a stratified pair of congruences such that for one of the congruences the given surface is a focal surface. This depends on four arbitrary functions of one variable. Then the problem is solved of the degree of arbitrariness with which it is possible to connect with a given congruence a second congruence such that they form a stratified pair. There are two cases. In the first case the congruence belongs to a linear complex, and the solution depends on a system of equations in involution, of which the general solution depends on two arbitrary functions of one variable. In the other case we find that the second congruence of the pair depends on one arbitrary function of one variable. The pair degenerates into a W-congruence with ruled focal surfaces counted twice.

The results are reached by means of Cartan's ω-notation. Reference is made to S. P. Finikov [Proektivno-Diferencial'naya Geometriya, Moscow, 1937, pp. 183–191]. *D. J. Struik* (Cambridge, Mass.).

Charrueau, André. Sur des congruences de droites ou de courbes déduites d'une surface. C. R. Acad. Sci. Paris **225**, 620–622 (1947).

Some results, stated without proof, are added to those of two papers [same C. R. **221**, 274–276 (1946); Bull. Sci. Math. (2) **70**, 127–148 (1946); these Rev. **7**, 262; **8**, 531] concerned with the study of congruences associated with a given surface and having planes as mean surfaces. *V. G. Grove* (Río Piedras, P. R.).

Watson, A. G. D. On the geometry of the wave equation. Proc. Cambridge Philos. Soc. **43**, 491–505 (1947).

To a rotation about the origin of Cartesian coordinates in 3-space there corresponds a set of four parameters, the coordinates of a point on the unit hypersphere in Euclidean 4-space. The effect of combining two rotations can be conveniently described in terms of certain 4th order E-matrices, similar to though not precisely the same as Eddington's E-numbers. The notation thus established makes it possible to define a "frame-vector," corresponding to an absolute set of axes, and to develop the transformation theory of frame-vectors. It is shown that transformations of Galilean coordinates in space-time can be dealt with along similar lines. The notation and ideas are used to describe certain geometrical properties of Dirac's wave equation and of analogous equations in 3 dimensions. *A. G. Walker.*

Abramescu, N. Une application de la géométrie affine à la théorie des courbes planes. C. R. Acad. Sci. Roum. **7**, 20–22 (1945).

The locus of centers of the osculating conics to a plane curve C is the affine evolute (envelope of affine normals). The author seeks those curves C whose affine evolutes are conics. He computes the affine curvature k of the evolute. By setting $k=0$ (parabola) a Riccati equation arises which is solvable explicitly in terms of Bessel functions. When $k=$constant$\neq 0$ (ellipse, hyperbola) the determination of the curves C is shown to depend on a nonlinear differential equation of second order. The paper is marred by numerous typographical errors. *J. L. Vanderslice.*

Bompiani, E. Geometria degli spazi a connessione affine. Ann. Mat. Pura Appl. (4) **24**, 257–282 (1945).

This paper takes up two closely linked topics in the theory of affinely connected spaces: (1) the relationship of the associated spaces of differentials with the underlying space, and (2) normal coordinates, normal tensors and extensions. It tries to deal with them from a completely geometric view-

point. The first topic has long been a controversial one (Are the associated spaces tangent, osculating, or neither?) and the author spares no pains in trying to improve on former attitudes, particularly those of Cartan and Weyl. He makes much use of projective geometrical methods. This first part is in the nature of a philosophical critique. As regards item (2), the author by virtue of his conclusions in the first part is able to reconstruct geometrically the normal tensors and extensions of Veblen and T. Y. Thomas and the nonholonomic extensions of E. Bortolotti, thus clarifying their significance in no small degree. *J. L. Vanderslice.*

Dalla Volta, Vittorio. Sulla torsione di una varietà a tre dimensioni dotata di una connessione affine. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 271–282 (1946).

Les principaux résultats établis dans cette note sont les suivants. (1) Dans chaque point d'un espace à connexion affine (L_3) il y a ∞^1 "facettes" qui contiennent le vecteur de torsion qui leur est associé. Ces facettes enveloppent un cône du deuxième degré. (2) Pour chaque vecteur covariant il y a en général une direction dans laquelle le transport parallèle a lieu sans torsion (l'accroissement des composantes dû à la partie antisymétrique de Γ^i_{jk} est nul). La relation entre le vecteur covariant et la direction est une corrélation. L'auteur étudie ensuite en détail le cas où cette corrélation est singulière. Certaines des propriétés étudiées sont généralisés pour des L_n. *M. Haimovici* (Jassy).

Dalla Volta, Vittorio. Sulle connessioni affini asimmetriche. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **5**, 315–326 (1946).

Si L^i_{jk} sont les composantes de la connexion affine d'un espace, on définit, d'après Bortolotti [Ann. Mat. Pura Appl. (4) **8**, 53–101 (1930)] la connexion affine conjuguée. Soit P un point de L_n et P_1 infiniment voisin à P. Les deux directions en P parallèles respectivement dans les deux connexions à une même direction en P_1 sont en relation homographique. L'auteur examine les directions doubles de cette homographie. Il arrive aux résultats suivants. Les connexions semi-symétriques (dont la torsion a l'expression $T_{hk}{}^i=\delta_h{}^i\varphi_k-\delta_k{}^i\varphi_h$) sont caractérisées par le fait de donner lieu à une direction double PP_1 et à ∞^{n-2} directions doubles situées dans le plan défini par le vecteur covariant $T_{ih}{}^i$. Les coefficients de l'équation en S correspondante sont des puissances d'une forme linéaire. Pour qu'il y ait un hyperplan passant par PP_1 dont toutes les directions soient doubles, il faut et il suffit que le tenseur de torsion soit de la forme a^iT_{hk}. L'équation en S a alors ses derniers $n-1$ coefficients nuls. Ces questions sont ensuite appliquées au cas particulier $n=3$ et mises en relation avec les résultats obtenus dans un travail précédent [voir plus haut]. *M. Haimovici* (Jassy).

Ohkubo, Takeo. Geometry in a space with generalized metrics. II. J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **10**, 157–178 (1941).

L'auteur étudie les espaces d'éléments linéaires doués d'une métrique $ds^2=g_{ij}(x,x')dx^idx^j$. Le premier chapitre est consacré à l'introduction d'une connexion euclidienne à l'aide de six axiomes de nature intrinsèque. Cinq de ces axiomes sont les mêmes que ceux imposés par Cartan [Les espaces de Finsler, Actual. Sci. Indus., no. 72, Hermann, Paris, 1934] aux espaces de Finsler. Le sixième [numéroté 5° par l'auteur] est $\sigma^i=K^i_k\omega^k$ avec

$$\sigma^i=dl^i+\tfrac{1}{2}E^{ij}\{-g^{ls}g_{k;j;l}l^kl^s+g_{k;j}l^j-g^{ls}g_{k,j}+g^{ls}g_{j,k}\}dx^k;$$

$g=g_{ij}x'^ix'^j$, $g_k=g_{kj}x'^j$, $2G_{ij}=g_{ij;s}+g_{js;i}$; $E^{ij}G_{jk}=\delta_k{}^i$; $l^i=x'^i/g^{\frac{1}{2}}$; $K^i_kH^k_j=\delta^i_j$; $H^k_{\cdot k}=\delta_k{}^i+l^jg^{ih}C_{jk}$; $C_{ijk}=\partial g_{ij}/\partial x'^k$; $N_0=N_kl^k$. Cet axiome est satisfait identiquement dans les espaces de Finsler. [Dans un autre travail [Tensor **3**, 48–55 (1940); ces Rev. **2**, 23], en étudiant la même question, l'auteur avait introduit, au lieu de celui-ci, un autre axiome qui ne satisfait pas à cette dernière condition.]

Dans le deuxième chapitre est étudiée la théorie des courbes; dans le troisième la courbure et la torsion de l'espace et les identités de Bianchi. La méthode est celle de Cartan [loc. cit.]. Dans le dernier chapitre est étudiée la théorie des hypersurfaces: les formules fondamentales et les formules de Gauss et Codazzi, par des méthodes analogues à celles des espaces de Finsler [voir M. Haimovici, Ann. Sci. Univ. Jassy **20**, 39–58 (1935); H. Hombu, J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **5**, 67–94 (1936)]. *M. Haimovici* (Jassy).

Craig, H. V. On the structure of certain tensors. Math. Mag. **21**, 21–29 (1947).

The author's purpose is to express the Mth order intrinsic derivative of a vector in terms of contractions of extensors. In order to do this, he determines a vector by contraction of extensors ($V^{\alpha\alpha}$, $T^{\epsilon}_{\alpha\alpha}$) and expresses the intrinsic derivative of this vector in terms of these extensors. By introducing a fundamental extensor (whose components are the Kronecker delta and the Christoffel symbols multiplied by the tangent vector to the curve under consideration) and replacing the extensor $T^{\epsilon}_{\alpha\alpha}$ by the fundamental extensor, the author expresses the Mth order intrinsic derivative of the vector V^{α} in terms of contractions of $V^{\alpha\alpha}$ and the Mth extensive derivative of the fundamental extensor. The same problem is solved for the covariant vector V_{α} by a similar technique. *N. Coburn* (Ann Arbor, Mich.).

NUMERICAL AND GRAPHICAL METHODS

Neĭsuler, L. Ya. On the construction of k-fold tables for functions of n ($n\geqq 3$) variables with a minimum number of entries. Doklady Akad. Nauk SSSR (N.S.) **56**, 343–346 (1947). (Russian)

The author continues his series of articles on the problems of tabulating functions of many variables by means of auxiliary tables. [Cf. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1947**, 597–608 (1947); these Rev. **9**, 104.] This paper treats of functions of more than 6 variables and of the problem of minimizing the number of times that auxiliary tables have to be consulted. This is what is meant by "minimum number of entries" in the title. For functions of n variables of one of three special types this minimum is the greatest integer in $(3n-1)/2$. This situation is illustrated by a function of 9 variables. Three auxiliary tables suffice. Two of these are entered four times, the third thrice. *D. H. Lehmer* (Berkeley, Calif.).

Placzek, G. The functions $E_n(x)=\int_1^\infty e^{-xu}u^{-n}du$. Appendix A, by Dr. G. Blanch: An asymptotic expansion for $E_n(x)$. Appendix B: Table of the functions $E_n(x)$, prepared by the Mathematical Tables Project. National Research Council of Canada. Division of Atomic Energy. Document no. **MT-1**, 39 pp. (1946).

For the functions $E_n(x)$ as defined above which play an important role in diffusion theory, and which are also useful

in many other mathematical problems, a collection of formulae and expansions together with Fourier and Laplace transforms are given. Appendix A contains a new asymptotic expansion to be used for large $x+n$:

$$E_n(x)=\frac{e^{-x}}{x+n}\left\{1+\frac{n}{(x+n)^2}+\frac{n(n-2x)}{(x+n)^4}+\frac{n(6x^2-8nx+n^2)}{(x+n)^6}\right\}+R(n,x),$$

where $R(n, x)$ is specified. Appendix B contains extensive tables for E_n ($n=0(1)20$, $x=[0(.01)2;\ 7D]$, $[2(.1)10;\ 7\text{–}10D]$) and some auxiliary tables.

J. G. van der Corput (Amsterdam).

***Meyer zur Capellen, W. Mathematische Instrumente.** 2d ed. Akademische Verlagsgesellschaft, Leipzig, 1944; J. W. Edwards, Ann Arbor, Mich., 1947. x+313 pp. (1 plate). $10.50.

This second edition differs only by minor corrections and additions from the first [1941]. The book is practically a catalogue of the best commercial instruments, with many excellent photographs, which are quite clear even in the American reproduction, line diagrams and enough of the theory to satisfy the physicist and the engineer. Details of design have been stressed and the analysis of the mechanisms has been carried out with the purpose of indicating the sources of errors and the accuracies to be expected.

The introductory portion of the book is an analysis of the basic component mechanisms of which the instruments and mechanical computers are constructed. These include linkage mechanisms, two and three-dimensional cams, sliding bars, slide rules employing various types of function scales and gear drives. A brief section is devoted to electrical slide wire analogs of the mechanical units. A large section of the book is devoted to the desk type of digital computers, both the common stepping cylinder variety as well as the ones based on the pantograph principle. Their many mechanical features are well described and the different commercial designs compared. There is some discussion of various methods of computation with them.

Machines for the solution of sets of linear equations which are described include the Wilbur mechanical solver, the Bode electrical network solver and the Mallock transformer solver. Solvers of polynomials of higher degree include the Hart-Travis potentiometer network and the Dietzold-Mercner (Bell Telephone Co.) mechanical "Isograph." References are made to other types.

The drafting table instruments which are described include curve plotting aids in both rectangular and polar coordinates, transformation devices such as the pantograph and the affinigraph, references to many photogrammetric devices used for preparing maps from aerial photographs, devices for changing from one coordinate system to another. Linkage devices for drawing conic sections, cycloidal curves, spirals and other curves are described. Differentiating devices include reflection and prism deviation comparators. Integrators include the familiar wheel and disc type as well as the wheel and sphere type, polar and roller planimeters, multiple value planimeters for moment measurements or for other functional integration. Integraphs for plotting the values of the foregoing integrators are described.

Differential equations are soluble graphically by proper combinations of integrators and gearing mechanisms. These may range from simple machines employing only one or two integrators up to the complex differential analyzer and simulators. Examples covering the range of complexity are illustrated. The Mader-Ott, Coradi, Henrici-Coradi and Harvey-Amsler harmonic analyzers are described.

An understandable shortcoming of the book is the dearth of American machines and instruments. Another shortcoming is the absence of the recent work on large scale digital computers. The Germans had developed at least one such machine, but it was not described here. Nevertheless, it will remain a very useful book for the overwhelming class of mathematical workers who will never have access to, nor need for, such large machines. Also, electronic techniques are too new to have been included. The book is terminated by an excellent bibliography of over 300 modern references.

M. Goldberg (Washington, D. C.).

Gavrilko, G. Un nouveau conicographe. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnīĭ Zbirnik **2**, 107–108 (1940). (Ukrainian. French summary) [MF 16948]

Description of a simple instrument for drawing conics, consisting of jointed links and slides. *M. Golomb.*

Leibin, A. Polarograph. Nauk.-Doslid. Inst. Mat. Meh. Harkiv. Univ. Geometričnīĭ Zbirnik **2**, 109–114 (1940). (Ukrainian. German summary) [MF 16949]

Description of an instrument invented by the author, consisting of jointed links and slides, for drawing the polar curve C' to any given curve C for a given circle K. Here C' is defined as the envelope of the polars of the points of C with respect to K. *M. Golomb* (Lafayette, Ind.).

dePackh, David C. A resistor network for the approximate solution of the Laplace equation. Rev. Sci. Instruments **18**, 798–799 (1947).

The construction of two d-c boards is described, each consisting of approximately two hundred resistors, that represent Laplace's equation in rectangular and axially symmetric coordinates, respectively. *G. Kron.*

Vaničev, A. P. An approximate method for the solution of problems of heat conduction with variable constants. Bull. Acad. Sci. URSS. Cl. Sci. Tech. [Izvestia Akad. Nauk SSSR] **1946**, 1767–1774 (1946). (Russian)

The boundary value problem for nonstationary heat flow in composite solids in the case of coefficients that vary with temperature, i.e., where the differential equation becomes nonlinear, is a formidable one. The author replaces it by a difference equation problem which can be solved numerically. No attempt is made to show in what sense the solution of the difference equation is an approximation to the solution of the differential equation. The difference equation is to be considered as a reformulation of the physical problem. There is a discussion of the optimum choice of the time difference to be used. *M. Golomb.*

Placzek, G. The neutron density near a plane surface. National Research Council of Canada. Division of Atomic Energy. Document no. **MT-16**, i+7 pp. (2 plates) (1947).

In this paper the integral equation

$$\rho(z)=\tfrac{1}{2}\int_0^\infty \rho(z')E(|z-z'|)dz'=\Lambda\rho(z),$$

where $E(z)=\int_z^\infty e^{-zx}x^{-1}dx$, is solved by an iteration method. The iteration is started with a "trial function" of the form $\rho^{(0)}(z)=z+z_0+ae^{-\alpha z}$, where a and α are certain constants and z_0 has the known asymptotic value 0.71044609. The

constants a and α are then determined by the condition that the successive iterates agree in predicting the same (exact) boundary value $\rho(0)=3^{-\frac{1}{2}}$. Thus it is required that for the nth approximation $\rho^{(n)}(0)=\rho^{(n+1)}(0)=3^{-\frac{1}{2}}$, where $\rho^{(n+1)}(z)=\Lambda\rho^{(n)}(z)$ $(n\geqq 0)$. In this manner it is found that the constants appropriate for $\rho^{(0)}(z)$ and $\rho^{(1)}(z)$ are, respectively, $a_0=0.133096$, $\alpha_0=3.68962$ and $a_1=0.113536$, $\alpha_1=2.62032$. It is stated that the iteration process described yields solutions which are accurate to one part in a thousand.

S. Chandrasekhar (Williams Bay, Wis.).

ASTRONOMY

Lemaître, G. Les variables canoniques dans le mouvement képlérien. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 365–371 (1946).

The purpose of this article is to show that by a suitable choice of the determining function S the transformation from Cartesian coordinates and velocity components to the Delaunay variables may be made directly. This procedure avoids the usual detour of first transforming to the Jacobian elements. *D. Brouwer* (New Haven, Conn.).

Loweke, G. P. A method for synthesis of periodic orbits illustrated by undisturbed motion. Astr. J. **53**, 7–12 (1947).

Chazy, Jean. Sur les solutions d'Euler et de Lagrange du problème des trois corps. C. R. Acad. Sci. Paris **224**, 1677–1680 (1947).

The paper presents three remarks on the three-body problem: (a) on the homothetic nature of the rectilinear solutions; (b) on the equality of sides of the Lagrangian triangle; (c) on the variation of the length of a side of the triangle formed by the three bodies in the case of a homographic solution. *W. Kaplan* (Ann Arbor, Mich.).

***Lampariello, G. Sull'accelerazione del centro di gravità nel problema relativistico dei tre corpi.** Atti Secondo Congresso Un. Mat. Ital., Bologna, 1940, pp. 432–443. Edizioni Cremonense, Rome, 1942.

The author considers the relativistic correction to the motion of three bodies which, in the Newtonian case, move in a Lagrangian equilateral triangle solution. It is shown that, up to relativistic terms of first order, the acceleration of the center of mass of the system is a periodic function of time with mean value 0. *W. Kaplan.*

Clavo Rivera, Carlos A. About the regularization of the plane problem of three bodies. Revista Ci., Lima **49**, 3–69 (1947). (Spanish. English summary)

Continuing the work of García [same Revista **45**, 53–90, 91–133 (1943); these Rev. **5**, 17], the author carries the series computations to terms of higher degree and concludes that García's analysis of the orbits at collision is satisfactory. *W. Kaplan* (Ann Arbor, Mich.).

García, Godofredo. New forces which occur in our solar system in addition to universal gravitation. Univ. Nac. Tucumán. Revista A. **5**, 133–146 (1946). (Spanish)

This paper is essentially the same in content as a previous one [Actas Acad. Ci. Lima **8**, 3–6 (1945); these Rev. **6**, 190]. *W. Kaplan* (Ann Arbor, Mich.).

Mayot, Marcel. Sur la stabilité des amas d'étoiles. Ann. Astrophysique **8**, 43–130 (1945).

The investigation represents an extension of an earlier one by H. Mineur [same Ann. **2**, 1–6 (1939)]. A star cluster is considered as being equivalent to a homogeneous gaseous ellipsoid subject to the mutual gravitational attraction between the component stars and to the forces resulting from the general galactic field. In the first part of his paper, the author considers the secular stability of equilibrium configurations for clusters moving in circular paths around the galactic center. Stability can be generated, provided that the ratio between the intermediate and the largest axes of the ellipsoid is greater than 0.510 to 0.515; the ratio between the smallest and the intermediate axes appears to be immaterial. For a known galactic field of force, it is possible to delineate a family of equilibrium configurations, ranging between a sphere and the limiting stable ellipsoid. The condition that gives the lower permissible limit for the ratio between the intermediate and the largest axes of the ellipsoid is the same as the condition for the lower limit of the star density for a stable configuration.

In the final chapter of the paper the author considers stability problems for ellipsoidal clusters moving in noncircular orbits around the galactic center. In this case the ratios between the ellipsoidal axes will vary depending on the place of the cluster relative to the galactic center. If A, B, C are the coefficients in the expression $V=-\frac{1}{2}(Ax^2+By^2+Cz^2)$ for the internal potential of the cluster, and if α, β, γ are certain coefficients depending on the nature of the general galactic field and on the internal cluster motions, then there is shown to exist the following relation between the axes of the ellipsoid (a, b, and c): $(A+\alpha)a^2=(B+\beta)b^2=(C+\gamma)c^2$. This represents a generalization of H. Poincaré's relation for a fluid, isolated mass in rotation ($\gamma=0$) and of H. Mineur's relation applying to circular orbits only ($\beta=0$). The author investigates in some detail the case of near-circular motion around the galactic nucleus for clusters that have the shapes of ellipsoids of revolution. *B. J. Bok.*

Kumagai, Saizô. Zahlenmässige Bestimmung von ringförmigen Gleichgewichtsfiguren eines rotierenden Flüssigkeitstropfens. Proc. Phys.-Math. Soc. Japan (3) **25**, 578–581 (1943). [MF 15070]

The author gives numerical and graphical results of calculations for figures of equilibrium of a homogeneous fluid body rotating uniformly about an axis when the only forces acting are those of capillarity, the figures being ring-shaped (spheroidal figures having been adequately treated by others). *E. J. Moulton* (Evanston, Ill.).

Zeuli, T. Moto di una massa liquida omogenea e viscosa attratta da più centri lontani colla legge di Newton che assume forma prossima all'ellissoide. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **78**, 333–354 (1943). [MF 16247]

Continuing the work of Levi-Civita, Agostinelli, Pignedoli and others [cf., e.g., Pignedoli, same Atti Cl. Sci. Fis. Mat. Nat. **79**, 332–345 (1944); these Rev. **7**, 494], the author considers the internal motion of a liquid "planet" subject to internal and external gravitational forces and to internal viscosity. Solutions are sought such that the velocity vector is a second-degree function of the coordinates and the bound-

ing surface an ellipsoid. It is shown that the latter condition cannot be exactly fulfilled, but can be obtained to good approximation. *W. Kaplan* (Ann Arbor, Mich.).

Bhatnagar, P. L. Radial oscillations of a star. Bull. Calcutta Math. Soc. **38**, 168–176 (1946).

Finite oscillations are considered and the effect of amplitude on period and skewness of the oscillations is studied. The criterion assumed for a standing wave is that for any element of mass $r-a=r\eta(a)q(t)$, where r is the distance of the element from the centre at time t, a is that at an initial instant, and $q(t)$ is independent of the element considered. Such standing waves are sought for the three models introduced by Sterne [Monthly Not. Roy. Astr. Soc. **97**, 582–593 (1937)]. The results are not all consistent with the possible existence of a standing wave; in the reviewer's opinion this is due simply to the inadequacy of the standing wave criterion. *T. G. Cowling* (Bangor).

Herget, Paul. On the higher order effects of a differential rotation. Astr. J. **53**, 15–16 (1947).

Underhill, Anne B. Absorption lines formed in a moving atmosphere. Astrophys. J. **106**, 128–134 (1947).

MECHANICS

Kasner, Edward, and De Cicco, John. Transformation theory of physical curves. Proc. Nat. Acad. Sci. U. S. A. **33**, 338–342 (1947).

Consider a particle moving in a plane in a positional field of force. The system S_k consists of those curves along which constrained motion is possible with the pressure P and the normal component N of the force vector related by the equation $P=kN$. For k equal to 0, -2 and 1, S_k is the system of trajectories, the system of (generalized) brachistochrones and the system of (generalized) catenaries, respectively. Kasner has previously shown that the only point transformations (or contact transformations) which convert every S_0 into an S_0 (corresponding to another field of force) are the collineations. [E. Kasner, Differential-Geometric Aspects of Dynamics, Amer. Math. Soc. Colloquium Publ., v. 3_{II} (1913).] In the present note it is proved that the only point transformations (or contact transformations) which convert every S_k into an S_k, for $k\neq 0$, are the similitudes. *L. A. MacColl* (New York, N. Y.).

Signorini, A. Sopra alcune questioni di geometria delle masse, cinematica delle masse e stereodinamica. Ann. Mat. Pura Appl. (4) **24**, 1–11 (1945).

An axis l through the center of inertia G is a principal axis with respect to no point of l or to all points. For an axis l, not going through G, a "center of inertia" C is defined by means of the antipolar correspondence with respect to the ellipsoid of inertia. If l is a principal axis, it is a principal axis for the point C; if not, l can be a "right-hand" or a "left-hand" axis. Applications to the gyroscopic motions and to the motions of Staude and van der Woude. *O. Bottema* (Delft).

Gantmacher, F. R., and Levin, L. M. Equations of motion of a rocket. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 301–312 (1947). (Russian. English summary)

A rocket consists of three parts: (i) metallic components which remain unaltered during flight, (ii) a propellant which is consumed during the burning period and (iii) burning gases in the interior of the body. Eventually these gases pass out through the exit plane of the nozzle and are then no longer considered to form part of the body; while in the body their mass is negligible in comparison with the total mass, but they possess a high velocity relative to the parts (i) and (ii). By considering the linear and angular momentum of the systems S and Σ which consist of those parts of the rocket which at some fixed instant of time t belong respectively to (i), (ii) and (i), (ii), (iii), the authors derive the equations of motion of a rocket. In investigations of this type the chief difficulties arise from the fact that the familiar dynamical principles, which have been established by applying Newton's laws to rigid bodies of constant mass, are not necessarily true for systems of changing mass content. The equations as first obtained depend upon three general groups of forces, namely (a) external forces due mainly to the pressure of the air, (b) forces due to the reaction caused by the ejected gases and (c) Coriolis forces. The latter two groups are then investigated in greater detail and are expressed in terms of the rate of ejection of the gases from the burning charge surface and from the nozzle, the variation in density, the angular velocity and other parameters. The velocity of the centre of gravity relative to the body is also found and Euler's equations for the angular motion about the principal axes of inertia are derived and modified so as to allow for the rotation of these axes inside the rocket. *R. A. Rankin* (Cambridge, England).

Hydrodynamics, Aerodynamics

Todd, Olga. On some boundary value problems in the theory of the non-uniform supersonic motion of an aerofoil. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2141** (**8961**), 13 pp. (1945).

The author discusses the following two problems from the linearized theory of nonstationary two-dimensional supersonic flow. (A) Possio's problem: to find the velocity potential due to a two-dimensional airfoil in a supersonic stream whose shape and motion are prescribed. The solution was expressed by Possio in the form of an integral. The aim of the author is to give a rigorous discussion of Possio's proof, filling out certain gaps in his treatment. (B) This problem is the special case of Possio's problem where the motion of the airfoil is harmonic. The author discusses the solution by Temple and Jahn (who use Riemann's method) and by Possio (who obtains it by specializing the solution for problem (A)). *P. Lagerstrom* (Pasadena, Calif.).

Karpovich, E. A., and Frankl, F. I. Resistance of a delta wing in a supersonic flow. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 495–496 (1947). (Russian. English summary)

Puckett, A. E., and Stewart, H. J. Aerodynamic performance of delta wings at supersonic speeds. J. Aeronaut. Sci. **14**, 567–578 (1947).

Lawrence, H. R. The dynamics of a swept wing. J. Aeronaut. Sci. **14**, 643–650 (1947).

Evvard, John C., and Turner, L. Richard. Theoretical lift distribution and upwash velocities for thin wings at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1484**, 36 pp. (12 plates) (1947).

Jones, W. Prichard. Aerodynamic forces on wings in simple harmonic motion. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2026** (**8474**), 28 pp. (1945).

In an earlier report [same Rep. no. **2145** (**6711**) (1943)] a method is suggested for the calculation of the pressure distribution on a wing of any plan form in steady motion. The theory is now extended to include oscillatory motion. As it is based on vortex sheet theory, it is subject to the usual limitations. *Extract from the paper.*

Loitsiansky, L. G. Resistance of a grid of profiles in a flow of a viscous incompressible fluid. Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 449–458 (1947). (Russian. English summary)

The paper gives the results derived from approximate formulae for the resistance of an infinite plane grid and the loss of head in it. Results are based on the assumption of slight nonhomogeneity in the velocities at that cross-section of the wake where the boundary layers breaking away from neighbouring profiles merge. A comparison is carried out of the resistance of the grid profile with the corresponding Squire-Young formula for an isolated profile.

Author's summary.

Vinogradov, Yu. P., and Kufarev, P. P. On some particular solutions of the problem of filtration. Doklady Akad. Nauk SSSR (N.S.) **57**, 335–338 (1947). (Russian)

Lewy, Hans. Water waves on sloping beaches. Bull. Amer. Math. Soc. **52**, 737–775 (1946).

The problem investigated is the progressive wave on sloping beaches in two dimensions. Besides the usual assumption of incompressibility and zero viscosity, the only approximation introduced is the neglection of the second order terms in the boundary condition on the free water surface ($y=0$). The angle between $y=0$ and the bottom of the beach is restricted to $\beta=\pi p/2q$, p odd and relatively prime to q, and $0<p<2q$. This is then a more extended case than that of Miche [Ann. Ponts Chaussées **1944**, 25–78, 131–164, 270–292, 369–406; these Rev. **7**, 348] who considered the case $p=1$. The progressive wave can be analyzed into two standing waves with appropriate characteristics at large distances from the shore ($x=0$). The main feature of the treatment is the transformation of the potential problem and the boundary conditions into the solution of the following ordinary differential equation for the complex function $W(z)$: $f(d/dz)(id/dz-1)W(z)=E(z)$, where $x+iy=z=re^{i\theta}$ and $E(z)$ is regular within the sector $S_{2\beta}$ with $0>\theta>-2\beta$ with real part equal to zero on the boundary of $S_{2\beta}$ (with the possible exception of the origin). Here $f(\xi)$ is given by $f(\xi)\equiv 1$ for $q=1$,

$$f(\xi)=1+\sum_{n=1}^{q-1} i^n \xi^n \prod_{k=1}^{n}(1+\epsilon^k)/(1-\epsilon^k), \qquad q\geqq 2.$$

Out of this manifold of solutions $W(z)$ one has to pick the solutions which satisfy the boundary conditions at the free surface and the bottom together with certain conditions concerning the behavior of $W(z)$ near $z=0$ and $z=\infty$. The solution of this problem is given in closed form. The functions involved are exponential functions and exponential integrals of certain special types. *H. S. Tsien.*

Stoker, J. J. Surface waves in water of variable depth. Quart. Appl. Math. **5**, 1–54 (1947).

The author first considers the two-dimensional case of the so-called exact linear theory which neglects only the second order terms in the boundary condition on the free surface, but is otherwise exact. The angle β of the sloping beach is $\pi/2n$, where n is an integer. The method for obtaining the solution of a travelling wave towards the shore is the same as that used by H. Lewy [cf. the preceding review], who reduced the problem to that of solving an ordinary differential equation with constant coefficients. The complementary function is essentially connected with the standing wave which remains finite at the shore and the particular integral is essentially connected with the standing wave which has a singular point at the shore. The solution is simpler here than Lewy's due to the restriction on the angle of the beach. Graphs of solutions for $\beta=90°$, 45° and 6° are given. These show that the singular standing wave solutions deviate very little from their asymptotic values till about one wave length from shore and then the amplitude is 10 per cent less than the asymptotic values. These are in agreement with the results of Miche.

The author then gives a new derivation of the so-called shallow water theory. This derivation brings out very clearly the nature of the approximation which is essentially the neglection of second order terms in the depth of the water. The solutions in this theory for travelling waves are also given. These are then compared with those of the exact theory for the case $\beta=6°$. It is found that the difference is appreciable for points either too close to the shore or too far from the shore.

The paper concludes with a solution of the three-dimensional problem of oblique travelling plane wave incident upon a vertical cliff. The method of Lewy is extended to this case, and an explicit solution given.

In one of the appendices, the values of $E(z)=e^z\int_z^\infty t^{-1}e^{-t}dt$ are tabulated for $z=re^{i\beta_k}$, $\beta_k=e^{i\pi(1+k/15)}$, $k=1, \cdots, 7$.

H. S. Tsien (Cambridge, Mass.).

Charney, J. G. The dynamics of long waves in a baroclinic westerly current. J. Meteorol. **4**, 135–162 (1947).

Standard methods are used to investigate perturbations on a realistic model of the troposphere and lower stratosphere. This model permits vertical, but not horizontal wind shear. Friction is neglected. The wave velocity equation is identical with Rossby's equation if there is no vertical wind shear; if wind shear exists it still is a good approximation if the zonal wind is measured near 600 mb. The properties of the unstable waves are these: the meridional wind is 90° out of phase with pressure; lowest density occurs ahead of lowest pressures; vertical velocity is nearly in phase with meridional wind components. All the properties derived from the theory agree, at least qualitatively, with observations. *H. Panofsky* (New York, N. Y.).

Sverdrup, H. U. Wind-driven currents in a baroclinic ocean; with application to the equatorial currents of the eastern Pacific. Proc. Nat. Acad. Sci. U. S. A. **33**, 318–326 (1947).

From the equation of continuity and the equations of motion (in which accelerations and horizontal stresses are

neglected) an equation is derived which contains observable quantities only. These quantities are: wind stress, and a pressure function defined by $P=\int_z^d p\,dz$, where p is pressure, z depth, and d the level at which there is negligible stress. The two sides of the equation are evaluated separately from observations in the tropics and plotted against latitude. The agreement between the two sides of the equation is good, thus justifying the assumptions made in the theory.

H. Panofsky (New York, N. Y.).

Elasticity

Fichera, G. Sull'integrazione delle equazioni dell'elasticità. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 403–408 (1947).

Note préliminaire annonçant l'application de la méthode de Picone au calcul de l'équilibre d'un solide élastique étant données les déformations ou les forces au contour. Le concept directeur ne semble pas différer du procédé classique de Volterra et Lauricella [cf. Tedone, Enzykl. Math. Wiss. IV 211, chap. 24], le point essentiel résidant ici dans le développement de la solution en série d'un système fondamental de déplacements et de forces, qu'on forme au moyen des déplacements de Somigliana (solutions simples selon Love) et de leurs dérivées par rapport aux coordonnées du pôle.

B. Levi (Rosario).

Krzywoblocki, M. Z. On the so-called principle of least work method. J. Franklin Inst. **243**, 187–204 (1947).

An expository discussion of admissible functions in an application of direct methods of the calculus of variations in elasticity. The author indicates that the use of improperly chosen sets of approximating functions may lead to unreliable approximations. Numerical examples are included.

F. B. Hildebrand (Cambridge, Mass.).

Rukhadze, A. K. Influence of transverse force on torque in bending of a bar. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 351–356 (1947). (Russian. English summary)

The paper deals with the flexure of a long cylindrical beam clamped at one end and bent by a couple and a transverse force applied at the other end. Nonlinear strains and F. D. Murnaghan's stress-strain relations are used to obtain the solution satisfying the boundary conditions and the compatibility and equilibrium equations.

I. S. Sokolnikoff (Los Angeles, Calif.).

Targoff, Walter P. The associated matrices of bending and coupled bending-torsion vibrations. J. Aeronaut. Sci. **14**, 579–582 (1947).

Rosovsky, M. J. Application of integro-differential equations to dynamic problems of the theory of elasticity in case of after-effect. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 329–338 (1947). (Russian. English summary)

The author applies Volterra's integro-differential equation for oscillations of elastic bodies in the presence of after-effects to determine the forced transverse vibrations of an elastic rod, radial oscillations of a hollow elastic sphere and transverse vibrations of a thin simply supported rectangular plate. The solutions are obtained in the series of the form $\sum X_n(x, y)T(t)$.

I. S. Sokolnikoff (Los Angeles, Calif.).

Galin, M. P. Transverse oscillations of a plate. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 387–388 (1947). (Russian. English summary)

The transverse oscillations of an elastic plate, subjected to a normal load $p(x, y, t)$, are determined approximately by solving the variational equation

$$\int\int_R (D\nabla^2 w-\rho h\partial^2 w/\partial t^2-p)\delta w\,dx\,dy=0$$

by Galerkin's method. As an illustration free vibrations of a clamped rectangular plate are studied by choosing $w=c(t)(x^2-a^2)(y^2-b^2)$ and the results are found to be in excellent agreement with the known exact solution obtained by S. Iguchi [Ing. Arch. **8**, 11–25 (1937)].

I. S. Sokolnikoff (Los Angeles, Calif.).

Bijlaard, P. P. On the elastic stability of thin plates, supported by a continuous medium. Nederl. Akad. Wetensch., Proc. **49**, 1189–1199 = Indagationes Math. **8**, 737–747 (1946).

In this note the author discusses the problem of cylindrical buckling of thin plates supported by an elastic medium. He develops the relations between the work of Gough, Elam, and de Bruyne [J. Roy. Aeronaut. Soc. **44**, 12–43 (1940)], of van der Neut, and his own earlier work. [The problem was previously considered by the reviewer in Proc. Nat. Acad. Sci. U. S. A. **23**, 328–333 (1937).]

E. Reissner (Cambridge, Mass.).

Bijlaard, P. P. On the elastic stability of sandwich plates. I, II. Nederl. Akad. Wetensch., Proc. **50**, 79–87, 186–193 = Indagationes Math. **9**, 64–72, 149–156 (1947).

In the present two papers the author extends his work on cylindrical buckling [see the preceding review] of sandwich-type plates. He now treats the buckling of rectangular plates simply supported along two opposite edges with arbitrary boundary conditions along the other two edges. He also treats the buckling of an infinite plane strip acted upon by shearing forces along the edges.

E. Reissner.

Hamel, Georg. Aufbau einer Theorie der Häute und der dünnen Schalen nach der Methode von Lagrange. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1943**, no. 6, 27 pp. (1944).

The author makes use of the method of Lagrange multipliers in connection with the minimum principle for displacements to derive the differential equations and boundary conditions relevant to finite deformations of homogeneous isotropic elastic membranes and thin shells of uniform thickness. The multipliers are identified with stress-resultants and couples. In this work the formulation is referred to general curvilinear coordinates in the middle surface and is specialized, in particular, to the case when the coordinate curves coincide with the lines of curvature. With respect to shell theory the developments are made to depend upon Navier's hypothesis, which assumes that normals to the undeformed middle surface deform into normals to the deformed middle surface. While the effect of Poisson's ratio

is also neglected in the derivation, the author shows by invariance considerations that the effect of lateral contraction can be incorporated by suitably defining the elastic constants appearing in the final formulation. The presentation is self-consistent, in that it does not explicitly involve an averaging of the relations pertinent to the three-dimensional elastic continuum. *F. B. Hildebrand.*

Ilyushin, A. A. Stability of plates and shells beyond the proportional limit. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1116, 44 pp. (1947).

Translation of a paper in Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **8**, 337–360 (1944); these Rev. **7**, 43.

Zvolinskiĭ, N. V. Plane surface waves in an elastic half-space and their diffusion in a liquid layer. Doklady Akad. Nauk SSSR (N.S.) **56**, 363–366 (1947). (Russian)

Riznichenko, G. V. Transformation of media in geometrical seismology. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **11**, 311–333 (1947). (Russian. English summary)

The solution of typical direct problems and the interpretation of seismic prospection (inverse problem), in particular, the mapping of reflecting or refracting surfaces as deduced from observations, are much easier if the wave velocity W is constant, the space being isotropic and homogeneous with respect to W. The author studies the possibility of reducing the problems of geometrical seismology in an anisotropic and inhomogeneous Euclidean space A to the corresponding problems for an isotropic and homogeneous space $\bar{A}$, transform of A. In general $\bar{A}$ is non-Euclidean and the metric tensor $T(\bar{A})$ of $\bar{A}$ is given by the relation $T(\bar{A}) = V_0^2 W \cdot T(A)$. In some particular cases of two dimensional problems $\bar{A}$ is Euclidean and such cases may have practical importance. *E. Kogbetliantz.*

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Buxton, A. The relations between an incident ray and its emergent ray through an optical system. Philos. Mag. (7) **38**, 325–331 (1947).

The formulae in the first part of Buxton's paper are transformations of the basic formulae of Bruns-Hamilton. Unfortunately the author does not take into account that the derivatives $V_1, \cdots, V_4$ considered by him are functions of the variables in question. This makes the elimination of the variables L and M by the author's methods rather problematical as soon as more than a first order approximation is intended. For the solution of the problem in question the reviewer refers to his own papers [Trans. Amer. Math. Soc. **53**, 218–229 (1943); Quart. Appl. Math. **1**, 69–77 (1943); these Rev. **4**, 204].

In the second part, the author restricts himself to the approximation of the Seidel theory and obtains the third order coefficients of the characteristic function for the special case of a thick lens. *M. Herzberger.*

Toraldo di Francia, G. Il teorema di Malus-Dupin generalizzato e la trasformazione dell'interferenza inversa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 787–790 (1947).

Taking the coordinates x, y of the intersection point of the rays of a normal system with a plane as parameters, the author obtains for the direction cosines α, β, γ the condition that $\partial x/\partial\beta = \partial y/\partial\alpha$. He then considers an image formation which coordinates to any entering wave an emergent wave, so that there is a one-to-one correspondence between the rays through any point of the plane. This is equivalent to an optical image formation which images the plane in question sharply, and thus leads to the well-known cosine relation (1) $n\alpha' = \alpha + a$, $n\beta' = \beta + b$, where n is the ratio of the refractive indices of object and image space. However, a and b are constant only for the rays through a given point, and not the same for all rays of the space. This is easily seen if we take as a characteristic function E an arbitrary function of (x, y). That leads to (2) $\partial E/\partial x = n\alpha' - \alpha$, $\partial E/\partial y = n\beta' - \beta$. The above theorem is called by the author the generalized theorem of Malus-Dupin and has been applied to the problem of obtaining the connection between entering and diffracted wave. The authors' result, that the intensity of the diffracted wave does not depend upon the direction, has to be checked in view of the correction suggested above. *M. Herzberger* (Rochester, N. Y.).

Temmerman, Frits. Calcul numérique des rayons obliques. Rev. Optique **26**, 423–433 (1947).

Vašiček, Antonin. The reflection of light from glass with double and multiple films. J. Opt. Soc. Amer. **37**, 623–634 (1947).

The reflection of light from a plane glass surface coated with thin films is derived as follows. First, it is shown that the reflection from two parallel surfaces may be interpreted as the reflection from one fictitious equivalent surface. This result allows one to consider the reflection from three surfaces as a reflection from two and finally again as the reflection from one equivalent surface. By continuation of this process a method for the investigation of multiple systems of layers is obtained. Explicit formulae are given for the case of two thin films coated on glass and conditions for zero reflection are derived. Since the formulae for a greater number of layers become rather cumbersome simplified formulae are given by the author in which multiple reflections are neglected. The numerical test shows that this simplication is permissible in practical applications.

R. K. Luneburg (New York, N. Y.).

Tuckerman, L. B. On the intensity of the light reflected from or transmitted through a pile of plates. J. Opt. Soc. Amer. **37**, 818–825 (1947).

A new solution of the following problem is given. Parallel light is incident upon a pile of plane parallel plates. The plates are supposed to be identical in material and thickness. The thickness shall be great enough so that interference effects are negligible. To find the reflectance and transmittance of the pile if the reflection coefficients of the surfaces and the absorption coefficients of the plates are known. The author first gives the history of this problem and notes the presence of incorrect solutions in standard text and reference books. Then a simple recurrence formula is derived which allows one to determine the desired quantities if a new plate is added to the pile. This recurrence formula has constant coefficients and thus can be solved explicitly by the method of characteristic functions. The formal ex-

pression of the solution differs from a solution given by Stokes but is shown to be mathematically identical with it. The theory is developed under the assumption that there are infinitely many multiple reflections at each surface. The difference of this solution from the solution of the case of only a finite number of reflections is estimated in an appendix to the paper. *R. K. Luneburg* (New York, N. Y.).

Herpin, André. Calcul du pouvoir réflecteur d'un système stratifié quelconque. C. R. Acad. Sci. Paris **225**, 182–183 (1947).

The author shows that the reflection and transmission of electromagnetic radiation from a general stratified medium is given by a simple product of matrices provided that the incident electromagnetic wave is plane. The factors of the matrix product are determined by the constants ϵ, μ, σ and by the thickness of the nth layer, and can be expressed by a linear function of two Pauli matrices. Consequently, the product itself must be a linear function of the three basic Pauli matrices. This result can be interpreted as follows. A system of multilayers is equivalent to a system of two suitably chosen fictitious layers. The result provides a simple method of calculating the reflectance and transmittance of a stratified medium. *R. K. Luneburg* (New York, N. Y.).

Timofeyev, P. V., and Sorokina, V. V. On the electrostatic lenses field form. C. R. (Doklady) Acad. Sci. URSS (N.S.) **56**, 153–156 (1947).

The focusing action of an electrostatic field is discussed if the equipotential surfaces are second order surfaces of revolution. The best results are found in case of paraboloids. Though such fields cannot be realized physically it is possible to obtain fields whose equipotential surfaces have approximately parabolic shape. It is claimed that electron lenses built according to this principle gave better focusing results in all cases than the usual lenses used at present in electron optical instruments. *R. K. Luneburg* (New York, N. Y.).

Durand, Émile. Nouvelle représentation et forme plus générale des équations de l'électromagnétisme classique. C. R. Acad. Sci. Paris **225**, 567–569 (1947).

A generalization of Maxwell's field equations is obtained by noting that the Maxwell tensor satisfies the relation

$$\sum_r M_{pr}M_{qr} = |\Delta|^{\frac{1}{3}}\delta_{pq}.$$

[For the algebra of these quantities, see É. Durand, same C. R. **225**, 280–282, 375–377 (1947); these Rev. **9**, 75, 76]. It is shown that the equation so obtained reduces to Maxwell's case when the field potentials are real and satisfy the Lorentz condition. *C. Kikuchi* (East Lansing, Mich.).

Agostinelli, Cataldo. Sulla magnetizzazione di un cilindro di lunghezza finita in presenza di un campo magnetico qualsiasi. Ann. Mat. Pura Appl. (4) **20**, 221–255 (1941). [MF 16604]

Cotte, Maurice. Théorie de la propagation d'ondes de choc sur deux lignes parallèles. Rev. Gén. Électricité **56**, 343–352 (1947).

Assuming two parallel long wires above infinitely conductive ground, the conventional transmission line relations are stated replacing the ground by the image wires. Line 1 is excited at one end by a voltage to ground and induces on line 2 current and voltage waves by electromagnetic field coupling. The general solution in the case of weak coupling (so that perturbation methods apply) is shown to be a superposition of two oppositely traveling voltage waves and two oppositely traveling current waves, all mutually independent and with amplitudes which are determined by the terminal conditions. Several special cases are discussed. The most interesting example is termination into a capacitance; the solution is obtained by operational methods, particularly Carson's integral, and involves the Laguerre polynomials. Reference to experimental results obtained by A. Mauduit [same vol., 331–343] shows satisfactory agreements of numerical values and interesting oscillations not previously explained. The method is also applied to a single line terminated into a capacitance with similar results.

E. Weber (Brooklyn, N. Y.).

Shen, D. W. C. Generalized star and mesh transformations. Philos. Mag. (7) **38**, 267–275 (1947).

The n-terminal direct impedance equivalent of a star without mutual impedances is derived once again; the derivation is effectively that of Russell [cf. A. T. Starr, Electric Circuits and Wave Filters, London, 1938]. A similar result for the star with mutuals, too complex for brief description, is given. Finally conditions on direct impedances permitting the reverse transformation to a star without mutuals, which seem insufficient to the reviewer, are given along with the equations of transformation.

[Reviewer's note: the first result is ascribed to Rosen [J. Inst. Elec. Engrs. **62**, 916–918 (1924)] although it is a simple consequence of equivalent (b) in the list of equivalent networks given by G. A. Campbell [Trans. Amer. Inst. Elec. Engrs. **30**, 873–909 (1911)] and the particular result for 4 terminals had been given by Campbell in 1904 [Elec. World and Engr. **43**, 647–649]. Moreover the star with mutuals is equivalent (c) in the same list [loc. cit.], so there has been no question, as the author implies, of the existence of the transformation. Campbell's explicit formulas for the elements of the equivalent networks in terms of the network discriminant are given with great brevity and supply the interrelation required.] *J. Riordan* (New York, N. Y.).

v. Laue, M. Eindeutigkeitssätze in der Theorie der Supraleitung. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1946**, 86–88 (1946).

With the methods of potential theory uniqueness theorems are derived for the electrodynamics of the superconductor of F. and H. London [Proc. Roy. Soc. London. Ser. A. **149**, 71–88 (1935)]. It is shown that for a multiply connected isolated superconductor the field and current distribution are unique if the field at infinity and the moduli of the superconductor potential ψ are given. Here ψ is defined within the superconductor as a multivalued function by the relation $\operatorname{grad}\psi = A + \Lambda cJ$ (A, vector potential; J, supercurrent; Λ, characteristic constant of the superconductor).

F. London (Durham, N. C.).

Quantum Mechanics

Auluck, F. C. A note on transmission across potential barriers. Philos. Mag. (7) **38**, 289–291 (1947).

The author calculates the reflection coefficient of a particle penetrating through a plane strip in which the potential is a linear function of the distance from one of the planes bounding the strip. The potential outside of the strip is taken as constant on each side of it. *I. Opatowski.*

Costa de Beauregard, Olivier, et d'Espagnat, Bernard. Quelques remarques sur les paradoxes de la mécanique statistique classique. Rev. Questions Sci. (5) **8**, 351–370, 526–548 (1947).

Blokhinzev, D. A non-Hamiltonian method in the theory of elementary particles. Acad. Sci. USSR. J. Phys. **11**, 179–183 (1947).

This summarises an attempt to reach a compromise between the Heisenberg phase-matrix scheme and Hamiltonian methods. The latter demand an invariant time-ordering of "cause" and "effect," produce divergent results for point particles, and are incompatible with some aspects of the theory of extended particles: they attribute the interaction of heavy particles to the effects of a light particle field. Here, an "elementary scattering matrix" r is introduced to describe the absorption or emission of a light particle. The scattering matrix R determining the scattered waves ψ_s from the primary waves ψ_i by $\psi_s=R\psi_i$ is replaced by two matrices so that $\psi_s=R_{si}\psi_i+R_{ss}\psi_s$, R_{si} determining the relation between primary and scattered waves neglecting reaction of the scattered waves, and R_{ss} accounting for this reaction; R_{si}, R_{ss} are given in terms of r. Suggestions are made about a relativistically invariant cut-off factor, i.e., to introduce a "universal length" parameter s_0 to avoid divergences. This leads to a relation between s_0 and the mass of the heavy particle. *C. Strachan* (Aberdeen).

Blokhinzev, D. On a non-Hamiltonian method in the theory of elementary particles. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 266–271 (1947). (Russian)

A translation is reviewed above.

Wild, E. On first order wave equations for elementary particles without subsidiary conditions. Proc. Roy. Soc. London. Ser. A. **191**, 253–268 (1947).

It has been shown that the equations for particles of spin 0, $\frac{1}{2}$, 1 may be put in the form $\partial_\lambda\beta^\lambda\psi=i\kappa\psi$ such that it is not necessary to add subsidiary conditions and all of the equations of motion are obtained as quantum mechanical equations. The author therefore asks if this is possible for particles of higher spin. For integral spin it is shown that the only values of the spin which lead to positive total energy and require no subsidiary conditions are 0 and 1. For half-integral spin the only equations which may be quantized according to Fermi statistics without the introduction of an indefinite metric in Hilbert space are those of spin $\frac{1}{2}$. *H. C. Corben* (Pittsburgh, Pa.).

Heisenberg, W. Die beobachtbaren Grössen in der Theorie der Elementarteilchen. III. Z. Physik **123**, 93–112 (1944).

In connection with two previous papers [Z. Physik **120**, 513–538, 673–702 (1943); these Rev. **4**, 292] the author shows that the η-matrix, introduced by him in order to describe quantum mechanical collision processes without having recourse to certain "unobservable" quantities, also determines the stationary states of the system in question. The eigenvalues of η give the phase difference between the incident wave e^{-ikr} and the scattered wave e^{ikr}. If k is imaginary, $k=-i|k|$, then the "incident wave" becomes $e^{-|k|r}$ and the "scattered wave" $e^{+|k|r}$. The amplitude of the latter disappears for $\eta\to i\infty$. This is the condition for a stationary state. *F. London* (Durham, N. C.).

Davydov, A. S. On the theory of a particle with a spin 3/2. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 427–436 (1947). (Russian. English summary)

A spin-vector form of writing down equations for a particle with a spin 3/2 in the case when external fields are present is discussed in the paper. The case is considered of a particle having a zero stationary mass. The relation between the equations obtained and those proposed by Rarita and Schwinger is discussed. *Author's summary.*

Van Isacker, J. Sur les équations de particules de spin 3/2 et leur seconde quantification. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 262–267 (1947).

Starting with a variational form, the author writes the fundamental equations for a particle of spin 3/2 and defines the current vector and the impulse-energy tensor. Next, following the notation of a previous paper [same Bull. (5) **32** (1946), 554–563 (1947); these Rev. **9**, 71], the Lagrangian function, the current vector and the impulse-energy tensor are decomposed. Then the quantification relations for a particle of minimum mass are obtained, following a method of J. Géhéniau. *A. Schwartz.*

Flint, H. T. Energy in the nuclear field. Philos. Mag. (7) **38**, 22–32 (1947).

A summary is given of the chief results of the author's five-dimensional field theory [Proc. Roy. Soc. London. Ser. A. **185**, 14–34 (1940); these Rev. **8**, 554] in which the fifth coordinate is cyclic. In common with wave geometry [K. Morinaga, J. Sci. Hirosima Univ. Ser. A. **5**, 151–188 (1935)] it is assumed that microscopic phenomena may be described in terms of an operator line-element $d\sigma=\gamma_\mu dx^\mu$ ($\mu=1, \cdots, 5$), where the γ_μ are an anticommuting pentad of Dirac matrices. An application is made to the theory of a scalar nuclear field and a generalized expression for the energy is derived which includes both the field energy and the energy of interaction of the nucleon with the field. It is pointed out that the separation of these two energies is an arbitrary procedure, but one such procedure leads to the result that a nucleon may lose mass in interacting with a nuclear field. *H. C. Corben.*

Schroedinger, Erwin. Affine Feldtheorie und Meson. Actes Soc. Helv. Sci. Nat. **126**, 53–61 (1946).

Zilsel, P. R., Blair, J. S., and Powell, J. L. Effect of Pais' "f-field" on the symmetry of the meson field part of nuclear Hamiltonian. Physical Rev. (2) **72**, 225–228 (1947).

According to Breit, Thaxton and Eisenbud [same Rev. (2) **55**, 1018–1064 (1939)], the experimental data on proton-proton scattering at energies up to 2.4 Mev can be fairly well described by adding to the Coulomb interaction between the two protons a spherical potential well of radius $r_0=e^2/mc^2=2.8\times10^{-13}$ cm ("electron radius") and "depth" $D=11.3$ Mev, which is only a few percent smaller than the corresponding ^{1}S-potential between neutron and proton. The present authors examine the question of how this result would be modified if the Coulomb interaction e^2/r were replaced by $(e^2/r)(1-2\exp(-\kappa r/r_0))$, in accordance with the "f-field theory" of Pais [same Rev. (2) **68**, 227–228 (1945); cf. also Verh. Nederl. Akad. Wetensch. Afd. Natuurk. Sect. 1. **19**, no. 1 (1947); these Rev. **8**, 554]. The calculations show that, accepting r_0 as the radius of the well, the depth must be made velocity dependent: e.g., $D=7.6$ and 8.9 Mev for 0.4 and 2.0 Mev protons, respectively (with

$1/\kappa=2$). Moreover, these changes in the potential are sufficiently large to destroy the close equality of the specific nuclear proton-proton and neutron-proton force in a ^1S-state. An obvious conclusion is that the f-field theory is not compatible with, for instance, a symmetrical meson theory.
L. Hulthén (Lund).

Foldy, Leslie L. On the meson theory of nuclear forces. Physical Rev. (2) **72**, 125–130 (1947).

Besides the Møller-Rosenfeld-Schwinger mixture, other possible mixed meson theories exist in which the $1/r^3$- and $1/r^2$-singularities in the nuclear interaction are cancelled. Certain combinations of this nonsingular type, involving vector, pseudovector and pseudoscalar meson fields, lead to a finite dipole and central interaction of a very simple form in the limit where the meson masses connected with the different fields approach equality and the square of the coupling constant increases inversely as the mass difference. The obvious difficulties of interpretation (infinite coupling constants) are emphasized. Calculations on the two-nucleon problem with the vector-pseudovector combination (symmetrical version) indicate that satisfactory agreement with experimental results cannot be attained with this limiting form of the interaction, if the mass of observed cosmic ray mesons is employed. No significant improvement is obtained by choosing a more general combination, including pseudoscalar mesons, but substantially better agreement can be achieved with a meson mass of the order of 300 electron masses. Still the author concludes that the outlook for a satisfactory theory of nuclear forces based on the idea of evading singularities by mixing does not appear promising.
L. Hulthén (Lund).

Markov, M. The higher approximations of the perturbation theory in the case of scalar meson field interaction. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **17**, 661–666 (1947). (Russian)

Thermodynamics, Statistical Mechanics

***Frenkel, J. Kinetic Theory of Liquids.** Oxford, at the Clarendon Press, 1946. 488 pp. $13.00.

This is a very interesting book. Although already written in 1942 it is most seasonable and does a great service in bringing to the English reader a very unconventional point of view on the liquid state as it has been developed during the past two decades by Frenkel himself and others. It is not, as the title might imply, a systematic kinetic approach to the liquid state by rigorous methods of general statistical mechanics like the attempts of Mayer, Born and others which, however, have so far led only to generalities the physical significance of which is still quite obscure. The basis of Frenkel's theory is an essentially empirical model which approaches the liquid state from the side of the solid state. In most of the book the structure of a liquid is considered roughly as akin to that of a solid the long range order of which has been lost. Accordingly Frenkel starts with a study of the kinetics of lattice distortions, hole formation, diffusion of impurities and interstitial atoms, order and disorder transition, etc. [chapters I and II]. The theory of these processes is developed and discussed in full detail as a kind of introduction into the as yet unsolved problem of the phase transition into the liquid phase [chapter III]. That a book of this pioneer character naturally has a very personal style is certainly an asset even if this implies occasional unjustified criticism, as, for instance, of Ehrenfest's concept of second order transitions of which in spite of such dialectic exaggerations the author can, of course, not avoid making extensive use. The applications contain chapters [IV] on heat motion, diffusion, viscosity, hypersonic waves in liquids, [V] on orientation and rotational motions of molecules in liquid bodies, Debye's, Onsager's and Kirkwood's theory of the dielectric constant; [VI] on surface- and allied phenomena; [VII] on kinetics of phase transitions [with a disregard of contemporary efforts in other countries]; [VIII] on properties of solutions and high polymeric substances.
F. London (Durham, N. C.).

Leibfried, Günther. Über die Anwendbarkeit der Sattelpunktsmethode bei tiefen Temperaturen am Beispiel des idealen Bose-Gases. Z. Naturforschung **2a**, 305–310 (1947).

A recent paper of Schubert [same Z. **1**, 113–120 (1946); these Rev. **8**, 556] has shown that the method of steepest descents is not a good approximation for low temperatures and accordingly it was questioned whether the normal form of the Bose-Einstein distribution is still correct, in particular, for the case of strong degeneracy. In the present paper it is shown that in spite of the failure of the method of steepest descents the results derived by elementary methods by F. London [Physical Rev. **54**, 947–954 (1938)] remain valid.
F. London (Durham, N. C.).

Bogolubov, N. On the theory of superfluidity. Acad. Sci. USSR. J. Phys. **11**, 23–32 (1947).

The object of this paper is an attempt to withdraw from the "counterblast of objections" raised by L. Landau and others [same J. **5**, 71–90 (1941)] against the Bose-Einstein theory of liquid helium [L. Tisza, C. R. Acad. Sci. Paris **207**, 1035–1037, 1186–1189 (1938); F. London, Physical Rev. (2) **54**, 947–954 (1938)] and to adopt the point of view of the latter. By using the method of second quantization the author tries to show that the phenomenon of superfluidity can be explained on the basis of a theory of the degeneracy of a nonperfect Bose-Einstein gas of certain "quasi-particles" representing elementary excitations of a continuum. The author ignores the fact that there is no Bose-Einstein condensation for quasi-particles which, like the excitations, have no constant particle number. The result is obtained under the condition of certain approximations, neglecting second and higher order terms of a quantity called ϑ, which characterizes the non-commutable part of the quantized wave function. Although it might be justifiable to consider this quantity as small and the series in question as rapidly convergent it nevertheless appears insufficient to derive the entire absence of any interaction from a consideration of a first order approximation alone.
F. London (Durham, N. C.).

Verschaffelt, J. E. La diffusion des gaz et la thermomécanique. I, II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 372–395, 396–415 (1946).

Verschaffelt, J. E. Quelques remarques à propos de la mécanique statistique de Th. de Donder. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 416–426 (1946).

These papers are a continuation of the triangular discussion among Meixner [Ann. Physik (5) **39**, 333–356 (1941)], Géhéniau [same Bull. (5) **29**, 31–36 (1943); these Rev. **7**, 184], and the author [same Bull. (5) **28**, 455–489 (1942); these Rev. **7**, 184]. A principal result is that "M. Géhéniau a bien voulu reconnaître que ma critique du travail de Meixner est justifiée."
C. C. Torrance.